Mini Course on Game Theory[1]
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[1] Thanks to Debashis Pal for some of his slides.

Introduction

- *"The major pleasures of the social sciences stem from an elementary property of human beings: Man is capable of producing more complex behavior than he is capable of understanding."*[2]

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   - ToMM attributes thinking to others and attributes a ToMM to others. ("I can think about you thinking about me.")

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    - Signaling games (used car market, suicide)
    - Cheap talk games (stock recommendations, campaign promises)
  - Repeated games (price-fixing, Medieval Law Merchant)

Extensive Form: Perfect Information

- A *decision node* is a point in the game at which someone has a decision to make.
- Out of a decision node is a series of branches, where a *branch* represents an action available to the decision maker.
- An *outcome* is a series of actions or a path through the tree.
- Associated with an outcome is a *payoff* for each player.

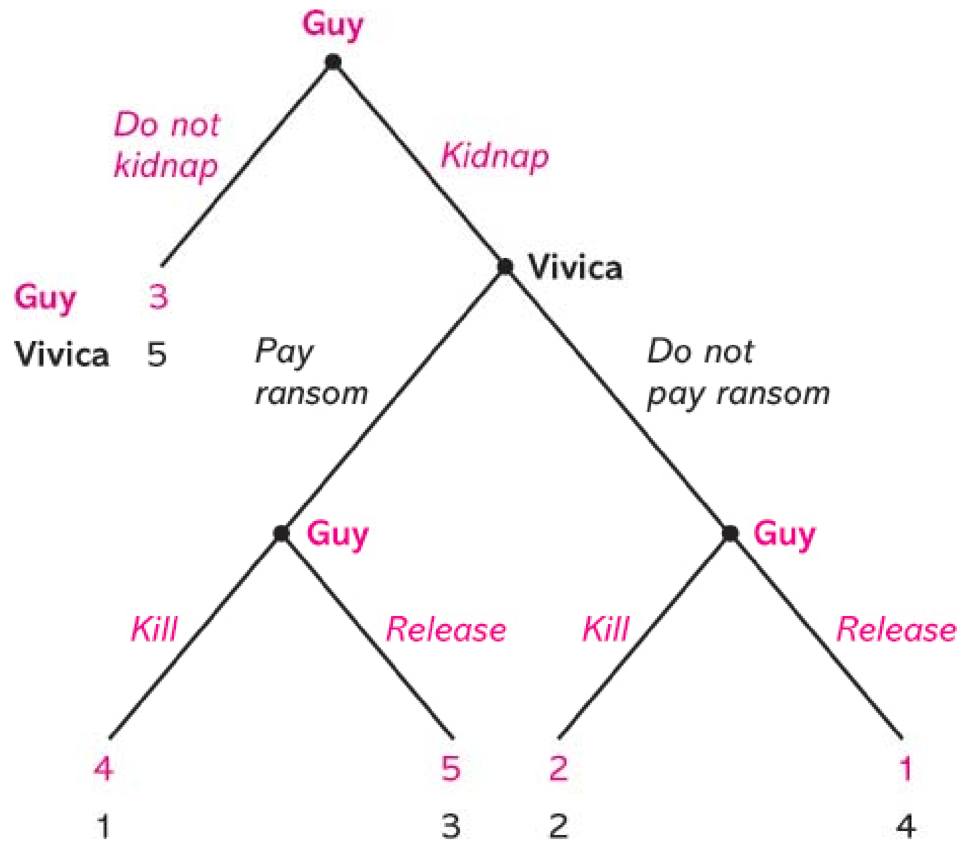


Figure 8.1 The Extensive Form of the Kidnapping Game (from Chapter 2)
Harrington: Games, Strategies, and Decision Making, First Edition

Extensive Form: Imperfect Information

- An *information set* comprises all of the decision nodes that a player is incapable of distinguishing among.
- If the information set has more than one node, then the player is uncertain as to where exactly he/she is in the game.
- In a perfect information game, each information set has exactly one node.
- In an imperfect information game, at least one information set has more than one node.

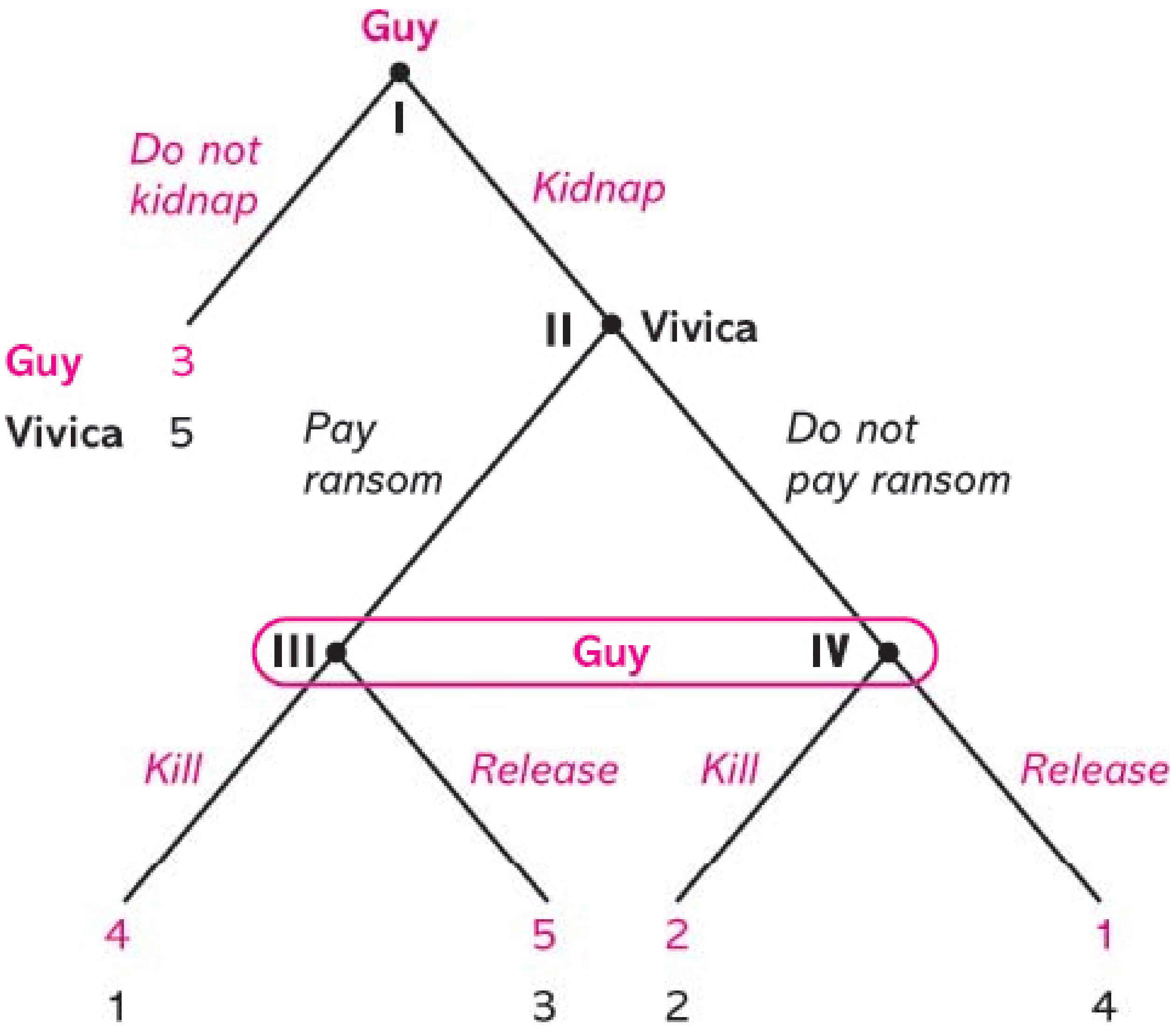


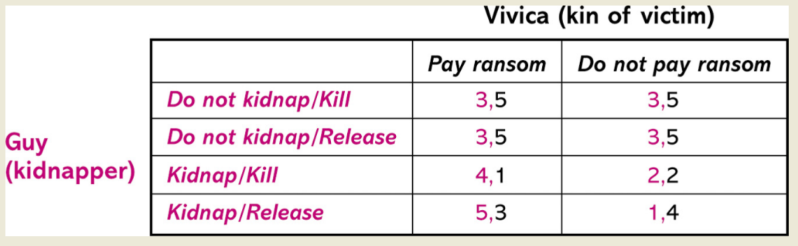
Figure 2.6 Kidnapping Game When the Exchange Is Simultaneous. The Box Around Nodes III and IV Represents the Information Set at the Point That Guy Has to Decide What to Do with Orlando
Harrington: Games, Strategies, and Decision Making, First Edition

Strategic Form

- A strategic form game describes:
  - who is making decisions?
  - over what they are making decisions?
  - how do players evaluate the outcomes?
- A strategic form game is defined by:
  - set of players
  - strategy set for each player
  - payoff function for each player which assigns a payoff to each strategy profile (which has one strategy for each player)

Strategic Form

- A *strategy* is a fully specified decision rule for how to play a game that incorporates every possible contingency.
- Going from the extensive form to the strategic form.



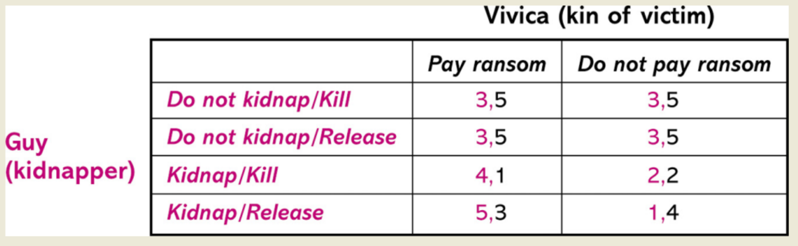
		Vivica (kin of victim)	
		Pay ransom	***Do not pay ransom***
Guy (kidnapper)	***Do not kidnap/Kill***	3,5	3,5
	Do not kidnap/Release	3,5	3,5
	Kidnap/Kill	4,1	2,2
	Kidnap/Release	5,3	1,4

Nash Equilibrium

- A strategy profile is a *Nash equilibrium* if each player's strategy maximizes his or her payoff, given the strategies used by other players.
- An equilibrium strategy for player j is both player j's decision rule and player i's conjecture as to player j's decision rule.
    - Let $s_j(i)$ denote what i believes j is going to play.
    - $(s_1, \ldots, s_n; s_2(1), \ldots, s_n(1), \ldots, s_1(n), \ldots, s_{n-1}(n))$ is a Nash equilibrium if:
        - ★ s_i maximizes player i's payoff given she believes player j will use $s_j(i)$, for all $j \neq i$, for all i (rationality)
        - ★ $s_j(i) = s_j$, for all $j \neq i$, for all i (accuracy of beliefs)
- Equilibrating process - How is a Nash equilibrium reached?
    - Introspection
    - Experience

Nash Equilibrium

- Nash equilibria



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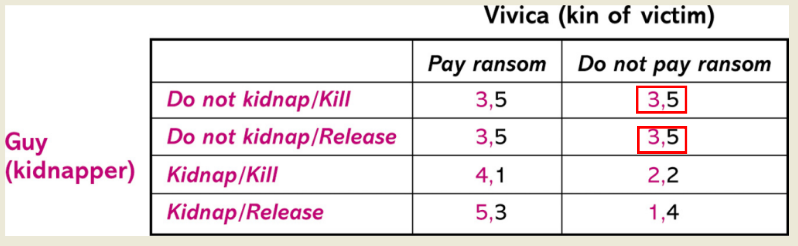
- Nash equilibria



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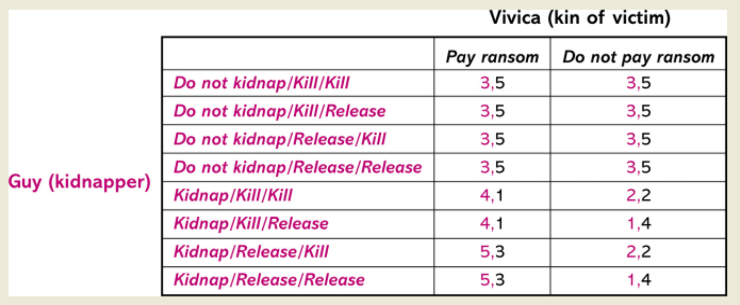

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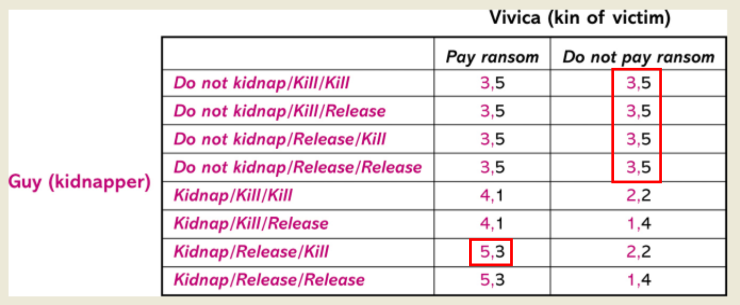

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Nash Equilibrium

Weakest Link: Airline Security

"Security is only as strong as its weakest link. And so if they get on in a foreign air carrier or if they target a foreign air carrier, it is going to be - could be a terrible tragedy again. So we need to make sure that there is a uniform raising of the level of security in a way that makes sense." Kenneth Quinn, Federal Aviation Administration counsel and chair of the Pan Am 103 task force.

Nash Equilibrium

Weakest Link: Airline Security

- Suppose there are $n \geq 2$ airlines and each has a strategy set of $\{1, 2, 3, 4, 5, 6, 7\}$. Think of a number as representing a level of security measures by an airline with a higher number indicating more effort put into security.
- Let s_i denote the strategy of airline i.
- Cost associated with security measures s_i is $10s_i$.
- Overall level of security is determined by the "weakest link" and, more specifically, is measured by

$$50 + 20 \min \{s_1, ..., s_n\}.$$

- Airline i's payoff is this common benefit less its personal cost:

$$50 + 20 \min \{s_1, ..., s_n\} - 10s_i.$$

Nash Equilibrium

Weakest Link: Airline Security

- Claim: At a Nash equilibrium, all airlines must choose the same level of security.
- To the contrary, suppose

$$s_i > \min\{s_1, ..., s_n\}$$

so airline i doesn't have the lowest level of security.

- Reducing its security by one unit from s_i to $s_i - 1$
    - leaves overall security level unaffected
    - reduces the airlines' cost by 10
- If there are any Nash equilibria, they must then be symmetric.

Nash Equilibrium

Weakest Link: Airline Security

- Suppose each airline chooses the same security measure, s'. An airline's payoff is then

$$50 + 20\min\left\{s', ..., s'\right\} - 10s' = 50 + 10s'.$$

- Airline 1's payoff from choosing more security $s'' > s'$,

$$\begin{aligned} 50 + 20\min\left\{s'', s', ..., s'\right\} - 10s'' &= 50 + 10s' - 10\left(s'' - s'\right) \\ &< 50 + 10s'. \end{aligned}$$

- Airline 1's payoff from choosing less security $s^0 < s'$,

$$50 + 20\min\left\{s^0, s', ..., s'\right\} - 10s^0 = 50 + 10s^0 < 50 + 10s'.$$

- It is an equilibrium for all airlines to choose s'.

Nash Equilibrium

Weakest Link: Airline Security

- Any symmetric strategy profile is a Nash equilibrium.
- Equilibria are Pareto-ranked with an equilibrium with more security being better. The payoff to an airline when all choose security measures s' is $50 + 10s'$ and this payoff is increasing in s'.

Nash Equilibrium

Weakest Link: Airline Security

- Experimental findings
- The payoff is exactly as specified above where it is measured in cents.
- There were between 14 and 16 subjects in a trial (so *n* is 14, 15, or 16).

Experimental Results for the Weakest Link Game

Action	Round 1 % of subjects	Round 10 % of subjects
7	31%	7%
6	9%	0%
5	32%	1%
4	16%	2%
3	5%	2%
2	5%	16%
1	2%	72%

Nash Equilibrium

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Network Effects: Operating Systems

- Suppose there are $n \geq 2$ people contemplating which operating system to purchase.
- Strategy set: $\{Mac, Windows\}$.
- Payoff to buying a Mac is $v + \alpha m$ where m is the number of people who choose *Mac* and $\alpha, v > 0$.
- Payoff to choosing *Windows* is αw where $w \, (= n - m)$ is the number of people who buy Windows.
- Noteworthy properties
  - Value of an operating system is higher when more people buy it, as measured by α.
  - Mac is intrinsically of higher value, measured by an amount v.

Nash Equilibrium

Network Effects: Operating Systems

- Nash equilibrium with Mac being the dominant system.
  - Consider the strategy profile in which everyone buys a Mac.
  - A consumer receives a payoff of
    - $v + \alpha n$ from buying a Mac
    - α from buying Windows.
  - Since $v + \alpha n > \alpha$ then it is optimal to buy a Mac.
- Nash equilibrium with Windows being the dominant system.
  - The payoff to buying Windows is αn, while the payoff from buying a Mac is $v + \alpha$.
  - If

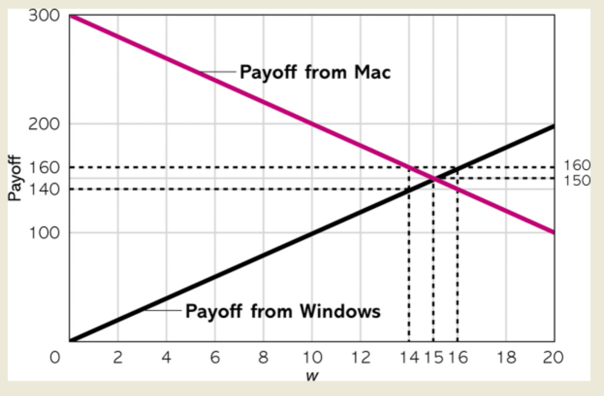
    $$\alpha n \geq v + \alpha \Rightarrow n \geq \frac{v}{\alpha} + 1$$

    then it is better to do like everyone else and buy Windows.
  - If the number of consumers is large enough then the advantage that Windows has from a bigger network effect exceeds the additional value due to the superior Mac technology.

Nash Equilibrium

Network Effects: Operating Systems

- In sum, there is always a Nash equilibrium in which all consumers use a Mac. If $n \geq \frac{v}{\alpha} + 1$ then there is also a Nash equilibrium in which all consumers use Windows.
- Example: $\alpha = 10$, $n = 20$, $v = 100$.



Nash Equilibrium

Network Effects: Operating Systems

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Nash Equilibrium

Network Effects: Operating Systems

- How do you generate common knowledge that a system will be dominant?
- Inform consumers that the system is great.
- Inform consumers that many other consumers are being informed that the system is great.
- Best generator of common knowledge in the U.S.?
- A commercial during the Super Bowl.

Nash Equilibrium

Network Effects: Operating Systems

Nash Equilibrium

Network and Congestion Effects

- Tipping
  - The more people who choose an option, the *more* attractive it becomes.
  - Results in symmetric equilibria in which different people act the same.
- Congestion
  - The more people who choose an option, the *less* attractive it becomes.
  - Results in asymmetric equilibria in which the same people act differently.
  - Examples: Applying for a position (job, college), which route to drive home, entry into a market

Nash Equilibrium

Network and Congestion Effects

- Restaurants
    - Yogi Berra: "No one goes there anymore because it's too crowded."
    - You don't want to be in a restaurant by yourself (network effect), but you don't want to have to wait in line for two hours (congestion effect).
- Gender and names[3]
    - Parents give their daughters male names which reduces the appeal of these names for sons.
    - Names that have gone (or are going) from male to female: Gail, Kim, Leslie, Sydney.
    - There can be a tipping point whereby once enough girls have what was previously a boy's name then it is no longer used as a boy's name.

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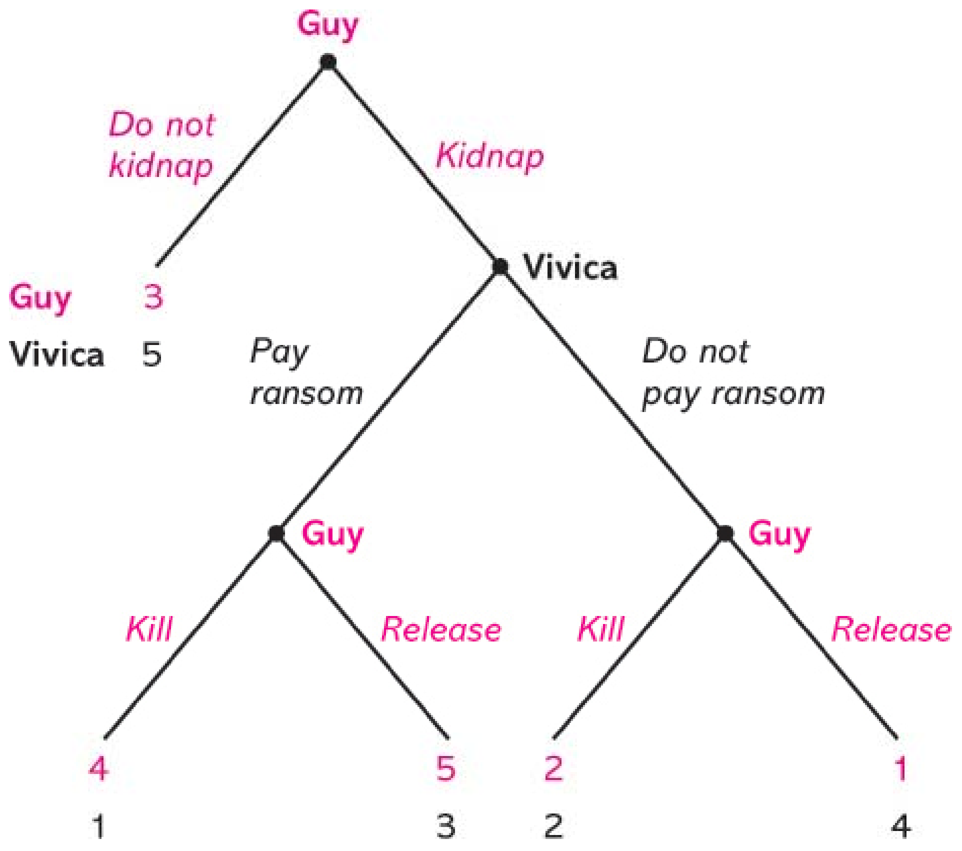
[3] Lieberson, Dumais, and Baumann, "The Instability of Androgynous Names: The Symbolic Maintenance of Gender Boundaries," *American Journal of Sociology*, 2000.

Backward Induction and Subgame Perfect Equilibrium

- For a game of perfect information, a strategy profile is a subgame perfect equilibrium if at each decision node, it assigns an action that maximizes a player's payoff.
- Note that the concept requires us to check that the actions are optimal even for information sets which are not reached by equilibrium play.
- Backward induction algorithm for finding subgame perfect equilibria for an extensive form of perfect information
    - For each of the final decision nodes, solve for optimal behavior.
    - For each of those decision nodes, replace the part of the tree beginning with that decision node with the associated payoffs.
    - Repeat the previous two steps until the initial decision node is reached.
- Existence of a subgame perfect equilibrium: in a game of perfect information, there is at least one subgame perfect Nash equilibrium.
- Every subgame perfect equilibrium is a Nash equilibrium, but not every Nash equilibrium is a subgame perfect Nash equilibrium.



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Harrington: Games, Strategies, and Decision Making, First Edition

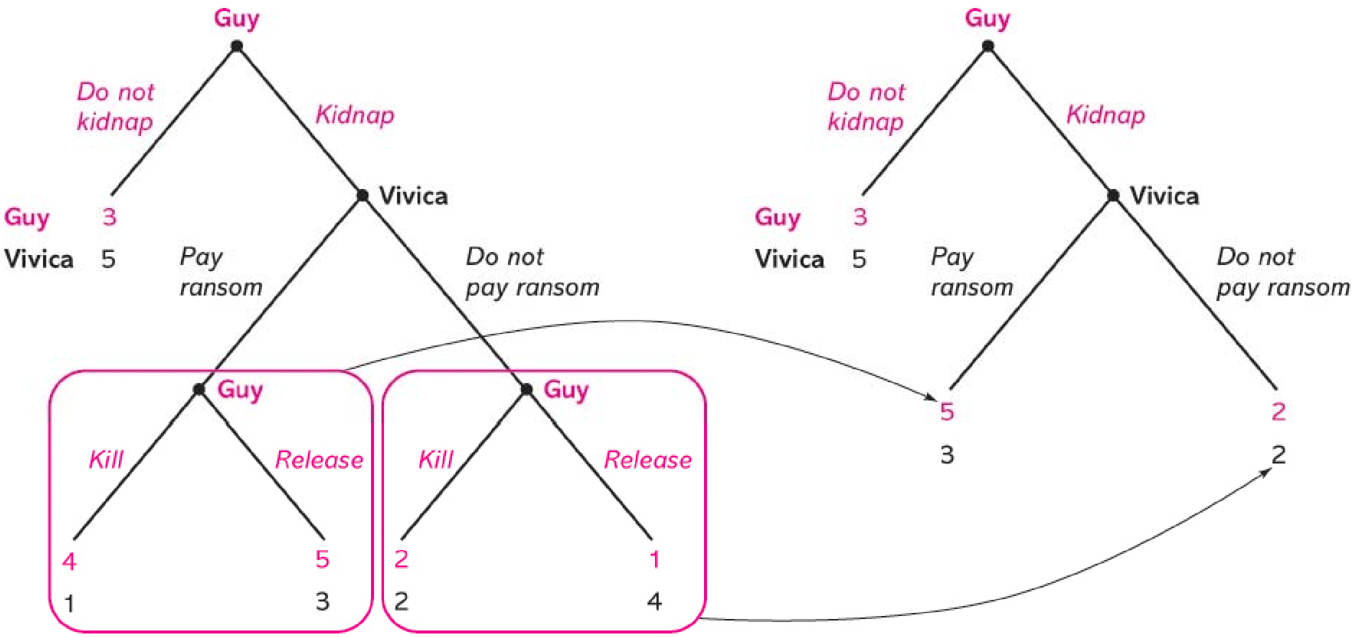
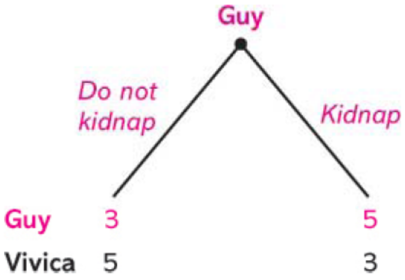


Figure 8.3 The Procedure of Backward Induction
Harrington: Games, Strategies, and Decision Making, First Edition

Backward Induction and Subgame Perfect Equilibrium



- Recall that there were five Nash equilibria but only one of them is a SPE:
  - Kidnapper: Kidnap/Release if ransom is paid/Kill is ransom is not paid
  - Kin: Pay ransom
- SPE rules out NE based on the incredible threat of the victim's kin not to pay ransom.

Backward Induction and Subgame Perfect Equilibrium

Bargaining

- Two players have to decide how to allocate a dollar.
- In an odd (even) period, player 1 (2) proposes an allocation. In response to which the other player can accept or reject. If he accepts, the game is over and the proposed allocation is made. If he rejects, the game moves to the next period.
- There are three periods and failure to agree by the end of the third period means loss of the dollar.
- At the end of each period, the pie shrinks proportionately by $1 - \delta$. The pie at the beginning of period 2 is δ and at the beginning of period 3 is δ^2. δ controls the cost of delay.
- Player's payoff equals the amount of money received.

Backward Induction and Subgame Perfect Equilibrium

Bargaining

- Period 1
  - Round 1: Player 1 proposes $x \in [0, 1]$ where x is the amount to be received by player 1 and player 2 is to get $1 - x$.
  - Round 2: Given x, player 2 chooses either *accept* or *reject*.
- Period 2 (reached if player 2 chose *reject* in round 2 of period 1)
  - Round 1: Player 2 proposes $x \in [0, \delta]$ where x is the amount to be received by player 1 and player 2 is to get $\delta - x$.
  - Round 2: Given x, player 1 chooses either *accept* or *reject*.
- Period 3 (reached if player 2 chose *reject* in round 2 of period 1 and player 1 chose *reject* in round 2 of period 2)
  - Round 1: Player 1 proposes $x \in [0, \delta^2]$ where x is the amount to be received by player 1 and player 2 is to get $\delta^2 - x$.
  - Round 2: Given x, player 2 chooses either *accept* or *reject*.

Backward Induction and Subgame Perfect Equilibrium

Bargaining

- x_t denotes the amount proposed for player 1 in period t.
- Period 3 game - Size of pie is δ^2.
  - Stage 2: Player 2 accepts all offers.
  - Stage 1: Player 1 proposes $x_3 = \delta^2$. Since all offers are accepted, optimal to set x_3 at its maximum of δ^2.
  - Equilibrium payoffs are $(\delta^2, 0)$.
- Period 2 game
  - Stage 2: Player 1 accepts iff $x_2 \geq \delta^2$.
  - Stage 1: Player 2 proposes $x_2 = \delta^2$ and receives a payoff of $\delta - \delta^2$.
    - ★ If $x_2 \geq \delta^2$ then the proposal is accepted and player 2's payoff is $x_2 - \delta^2$.
    - ★ If $x_2 < \delta^2$ then the proposal is rejected and player 2's payoff in period 3 is zero.

Backward Induction and Subgame Perfect Equilibrium

Bargaining

- Period 1 game
  - If agreement is not reached in period 1 then the equilibrium payoffs are $(\delta^2, \delta - \delta^2)$.
  - Player 2 accepts iff $1 - x_1 \geq \delta - \delta^2$. He gets a payoff of $\delta - \delta^2$ in period 2.
  - Player 1 proposes $x_1 = 1 - \left(\delta - \delta^2\right)$.
    - If she proposes $x_1 \leq 1 - \left(\delta - \delta^2\right)$, the offer is accepted and she earns x_1.
    - If she proposes $x_1 > 1 - \left(\delta - \delta^2\right)$ then the offer is rejected and her ensuing payoff is δ^2.
    - Since $1 - \left(\delta - \delta^2\right) \geq \delta^2 \Leftrightarrow 1 \geq \delta$ then she prefers to propose the maximum share that will be accepted by player 2.

Backward Induction and Subgame Perfect Equilibrium

Bargaining

- Subgame perfect equilibrium
  - Strategy for player 1
    - ★ In period 1, propose $x_1 = 1 - \delta(1 - \delta)$.
    - ★ In period 2, accept iff $x_2 \geq \delta^2$.
    - ★ In period 3, propose $x_3 = \delta^2$.
  - Strategy for player 2
    - ★ In period 1, accept iff $x_1 \leq 1 - \delta(1 - \delta)$.
    - ★ In period 2, propose $x_2 = \delta^2$.
    - ★ In period 3, accept iff $x_3 \geq 0$.
- Outcome
  - Player 1 proposes $1 - \delta(1 - \delta)$ in the first period and player 2 accepts. No delay in reaching an agreement.
  - Payoffs: $V_1 = 1 - \delta(1 - \delta)$, $V_2 = \delta(1 - \delta)$. Advantage to proposing first as $V_1 > V_2$.

Backward Induction and Subgame Perfect Equilibrium

Bargaining: Two-Period Version

- Suppose player 2 can commit herself not to be available for period 3.
- SPE strategy for player 1
  - In period 1, propose $x_1 = 1 - \delta$.
  - In period 2, accept iff $x_2 \geq 0$.
- SPE strategy for player 2
  - In period 1, accept iff $x_1 \leq 1 - \delta$.
  - In period 2, propose $x_2 = 0$.
- Outcome
  - Player 1 proposes $1 - \delta$ in the first period and player 2 accepts.
  - Payoffs: $V_1 = \delta$, $V_2 = 1 - \delta$.
  - Player 2's payoff has increased from $\delta(1 - \delta)$ to $1 - \delta$.
- *Eliminating options can be valuable in a strategic context.*

Backward Induction and Subgame Perfect Equilibrium

Bargaining: Fine Arts Auction Houses

Sotheby's

CHRISTIE'S

Dede Brooks

Alfred Taubman

Christopher Davidge

Anthony Tennant

David Boies

Backward Induction and Subgame Perfect Equilibrium

Bargaining: Fine Arts Auction Houses

- Judge Louis Kaplan was presiding over the consolidation of 38 civil lawsuits and auctioned off the position of lead counsel for plaintiffs.
- The law firm submitting the highest bid would be the lead counsel and receive payment equal to 25% of the amount by which the final settlement exceeded its bid. If the settlement was below its bid then it would not be paid and the entire settlement would go to the plaintiffs.
- Boies, Schiller & Flexner submitted the highest bid of $405 million (average bid was $130 million).
- Far exceeding the initial estimates, the final settlement was $512 million; the lead counsel made $26.75.
- The bargaining position of the lawyers were changed with this incentive scheme since they had no reason to agree to any amount below $405 million as they were no better off from doing so.

Backward Induction and Subgame Perfect Equilibrium

Bargaining: Fine Arts Auction Houses

- Modify bargaining game to where an agent acts on a person's behalf and his payoff is

$$\begin{cases} 0 & \text{if } x < x' \\ .25\left(x - x'\right) & \text{if } x \geq x' \end{cases}$$

- It is credible to reject all offers with $x \leq x'$.
- *Committing yourself to act in a manner different from your best interests can be valuable in a strategic context.*

Backward Induction and Subgame Perfect Equilibrium

Bargaining

- The Ultimatum Game: Experimental Evidence
- One period of bargaining.
- Assuming a player's payoff is monotonic in money, there are two SPE:
  - Proposer proposes that she receives the entire amount and the responder accepts.
  - Proposer proposes that she receives the entire amount less the unit of exchange and the responder accepts.
- Experiments conducted in the Slovak Republic in 1994
  - Amounts ranged from 60 Sk to 1500 Slovak crowns (Sk), where the average monthly earnings was 5,500 Sk.

Backward Induction and Subgame Perfect Equilibrium

Bargaining

Experimental Results for the Ultimatum Game

% requested by proposer	% offered (60 Sk)	% rejected (60 Sk)	% offered (1500 Sk)	% rejected (1500 Sk)
$<$ 50	6.3 (15)	6.7 (1)	7.2 (18)	6.7 (0)
50	28.7 (69)	0.0 (0)	30.8 (77)	1.3 (1)
50.5 - 60	46.3 (111)	17.1 (19)	38.4 (15)	4.0 (4)
60.5 - 70	15.9 (38)	42.1 (16)	12.4 (31)	6.5 (2)
More than 70	2.9 (7)	71.4 (5)	11.2 (28)	53.6 (15)
All offers	100.0 (156)	25.6 (40)	100.0 (250)	8.0 (22)

- "% Offered" indicate the percentage of proposals with a given offer.
- "% Rejected" give the percentage of offers that were rejected.

Backward Induction and Subgame Perfect Equilibrium

Bargaining

Experimental Results for the Ultimatum Game

% requested by proposer	% offered (60 Sk)	% rejected (60 Sk)	% offered (1500 Sk)	% rejected (1500 Sk)
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- “% Rejected” give the percentage of offers that were rejected.

Backward Induction and Subgame Perfect Equilibrium

Bargaining

Experimental Results for the Ultimatum Game

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- "% Offered" indicate the percentage of proposals with a given offer.
- "% Rejected" give the percentage of offers that were rejected.

Backward Induction and Subgame Perfect Equilibrium

Bargaining

- Theory predicts that all proposals would be close to 100%, in fact the proposer asked for no more than 70% in 97.1% of proposals in experiments with 60 Sk and 88.8% of proposals for experiments with 1,500 Sk.
- Theory predicts that responders accept all proposals that give them a positive amount. Quite to the contrary, rejection occurred very frequently. It is interesting, however, that rejection rates were lower when the amounts involved were higher. And, taking advantage of that expected behavior of responders, proposers proposed allocations more skewed in their favor.
- Rather than refuting backward induction, the contradictory evidence amassed by the Ultimatum trials may be due to misspecified payoffs.
  - People may care about the distribution of income.
  - Responders may reject out of spite.
  - Proposers may propose more equitable allocations to avoid invoking spite.

Games of Incomplete Information

Munich Agreement

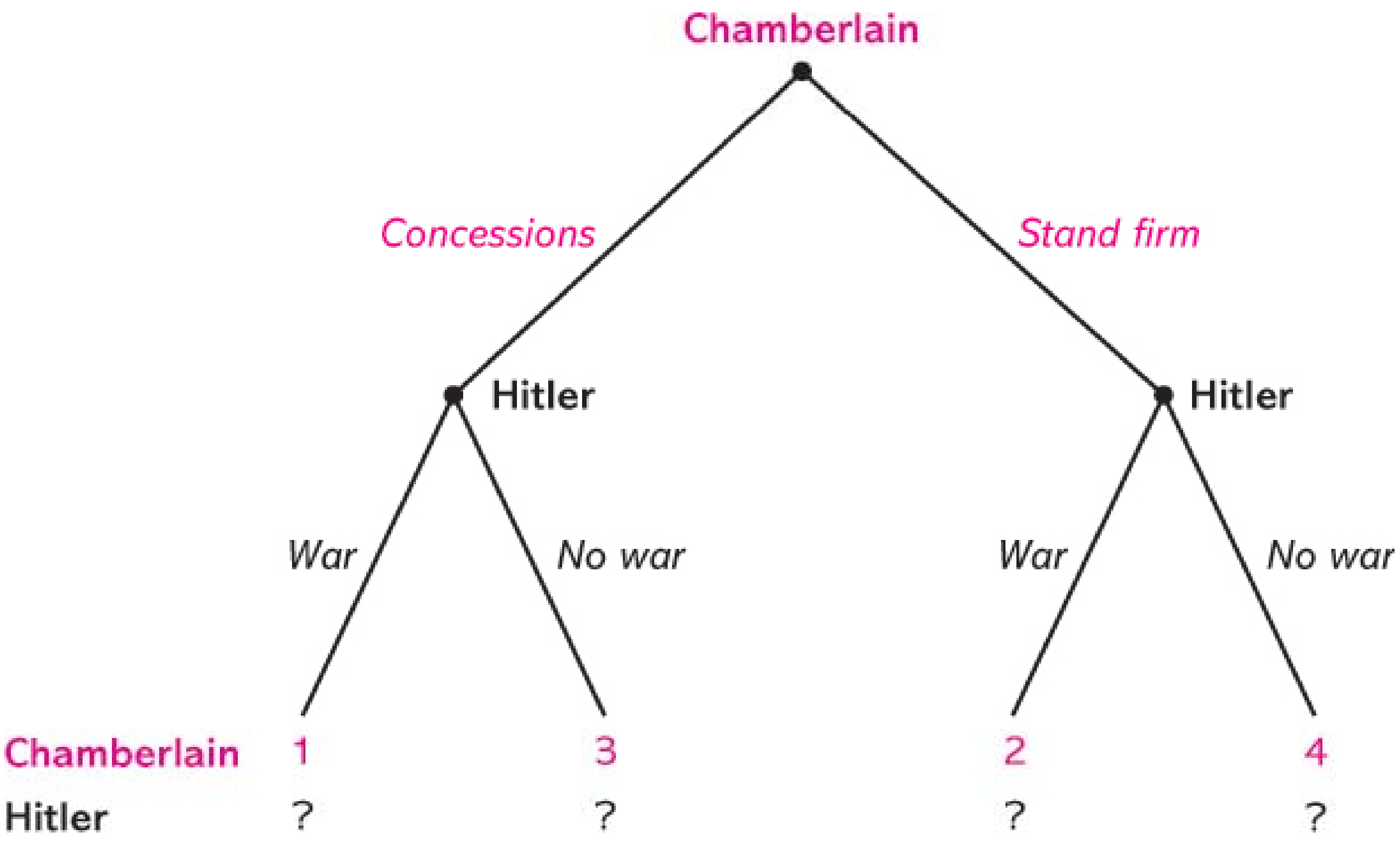


Figure 10.1 The Munich Agreement Game with Unknown Payoffs for Hitler
Harrington: Games, Strategies, and Decision Making, First Edition

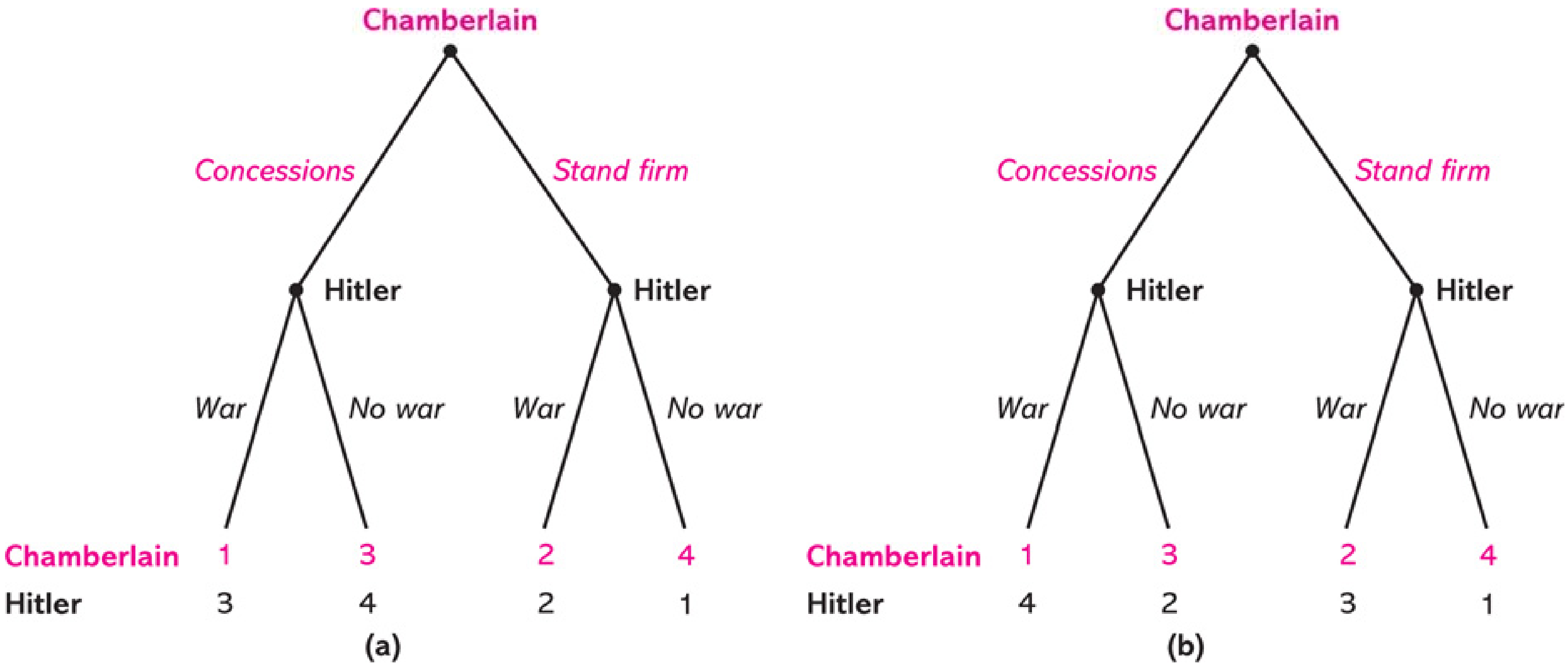


Figure 10.2 (a) The Munich Agreement Game when Hitler is Amicable.
(b) The Munich Agreement Game when Hitler is Belligerent
Harrington: Games, Strategies, and Decision Making, First Edition

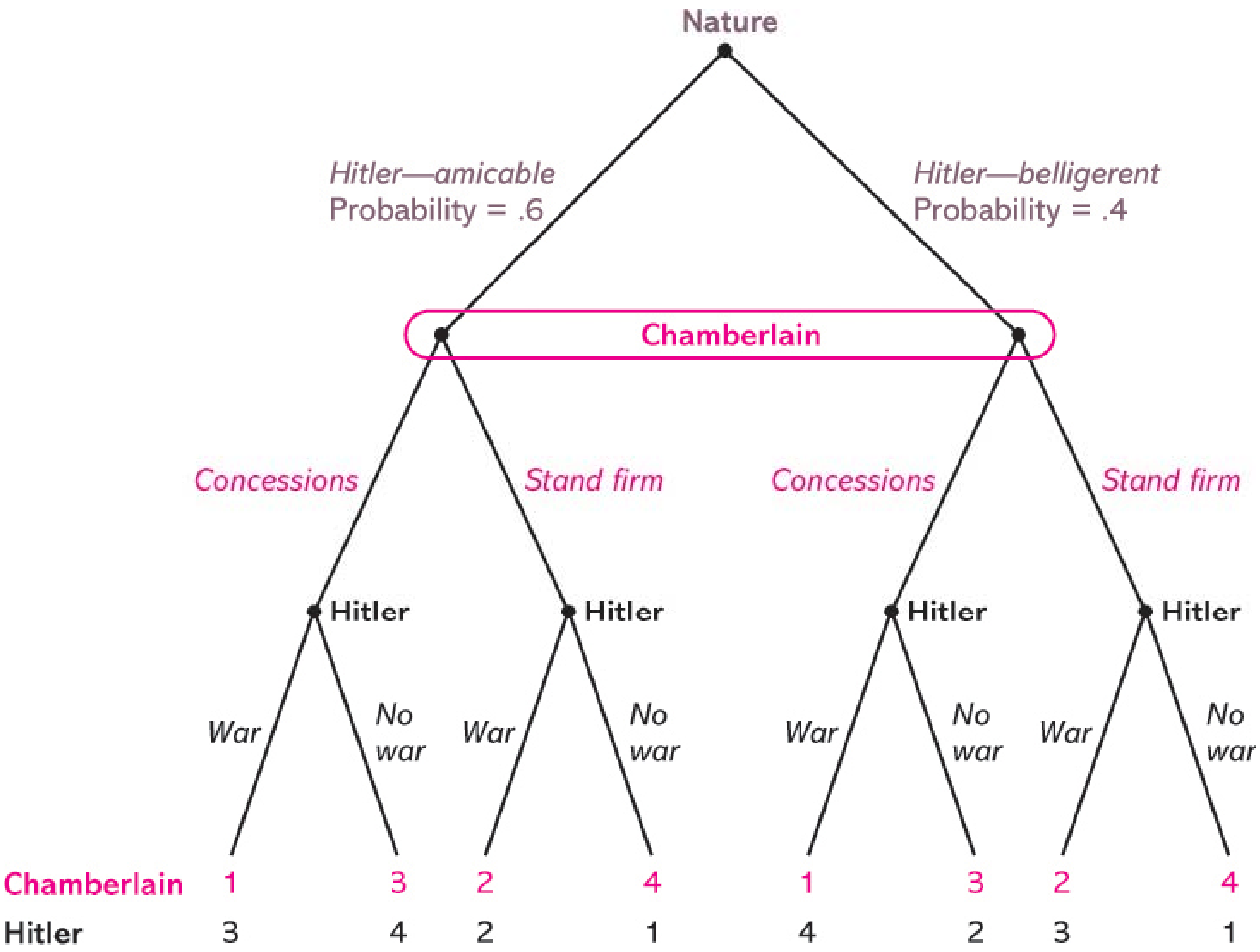


Figure 10.3 The Munich Agreement Game when Nature Determines whether Hitler is Amicable or Belligerent
Harrington: Games, Strategies, and Decision Making, First Edition

Games of Incomplete Information

Defining a Bayesian Game

- Space of types
  - What is private information to a player is referred to as the player's *type* and the collection of feasible types is the *type space*.
  - Typically, a player's type is his payoffs in which case the type space is the collection of possible payoffs for that player.
  - Munich Agreement: The Hitler type space includes two payoff configurations which are labelled *amicable* and *belligerent*.
- Probabilities over the type space
  - "Turn back the clock" to when a player did not know his type.
  - At that stage, random forces (Nature) determine each player's type.
  - A probability is assigned to each type which measures the likelihood of Nature choosing that type for a player.
  - *Common prior assumption* is that all players have a common prior set of beliefs over players' types.

Games of Incomplete Information

Defining a Bayesian Game

- Strategy sets for the player with private information
  - A strategy is conceived as being selected prior to when Nature moves.
  - A strategy states what to do given Nature's choice as to the player's type and whatever else a player may know (with regards to what actions have been chosen by other players).
- Solution Concept: Bayes-Nash (or Bayesian) equilibrium.
  - A strategy profile is a *Bayes-Nash equilibrium* if it prescribes optimal behavior for each and every type of a player given the other players' strategies, and does so for all players.
  - Consider a Bayesian game and let m_i denote the number of types of player i.
    - ★ For player i's strategy to be part of a Bayes-Nash equilibrium, it must be optimal for each of those m_i types.
    - ★ A strategy profile must then satisfy $m_1 \times m_2 \times \cdots \times m_n$ conditions to be a Bayes-Nash equilibrium as each player's strategy must be optimal for each of that player's possible types.

Games of Incomplete Information

Munich Agreement: Bayes-Nash Equilibrium

- A strategy for Hitler is a 4-tuple of actions as he has four informations sets according to what has been done by Nature and Chamberlain. A strategy for Hitler maps

$$\{(amicable,\ concessions),\ (amicable,\ stand\ firm),$$
$$(belligerent,\ concessions),(belligerent,\ stand\ firm)\}$$

  into

$$\{war,\ no\ war\}.$$

- Hitler's optimal strategy.
  - If *amicable* then choose *no war* when Chamberlain offers concessions and *war* when he stands firm.
  - If *belligerent* then choose *war* regardless of what Chamberlain does.
  - Optimal strategy is (*no war*, *war*, *war*, *war*).

Games of Incomplete Information

Munich Agreement: Bayes-Nash Equilibrium

- A strategy for Chamberlain is a single action as he has only one information set.
- Chamberlain's expected payoff from
  - concessions is
$$.6 \times 3 + .4 \times 1 = 2.2.$$
  - standing firm is
$$.6 \times 2 + .4 \times 2 = 2.$$
- Chamberlain's optimal strategy is to offer concessions.
- Bayes-Nash equilibrium:
  - Chamberlain offer concessions.
  - Hitler avoids war if he is *amicable* and goes to war if he is *belligerent*.

Games of Incomplete Information

Munich Agreement: Bayes-Nash Equilibrium

Games of Incomplete Information

First Price Sealed Bid Auction: IPV

- Assume there are $n \geq 2$ bidders and let v_i denote how much bidder i values the good.
- If bidder i wins the item and pays a price p then her payoff is $v_i - p$.
- Each bidder's value is a random draw from the interval $[0, 1]$ according to a uniform cumulative distribution function:

$$F(v) = \begin{cases} 0 & \text{if } v < 0 \\ v & \text{if } 0 \leq v \leq 1 \\ 1 & \text{if } 1 < v. \end{cases}$$

- A bidder's valuation is private information and, according to the common prior assumption, it is common knowledge that each bidder's value is drawn from $[0, 1]$ according to a uniform distribution.

Games of Incomplete Information

First Price Sealed Bid Auction: IPV

- Extensive form
    - Stage 1: Nature simultaneously chooses valuations for the n bidders and reveals v_i only to bidder i.
    - Stage 2: Bidders simultaneously submit bids.
    - Bidder with the highest bid wins the item and pays a price equal to her bid.
- Candidate symmetric Bayes-Nash equilibrium strategy:

$$b^*(v) = \left(\frac{n-1}{n}\right) v.$$

Games of Incomplete Information

First Price Sealed Bid Auction: IPV

- Bidder i's expected payoff is

$$\pi_i(b_i, v_i) = Prob(b_i > b_j \text{ for all } j \neq i) \times (v_i - b_i).$$

- Given bidder j uses the bidding rule $\left(\frac{n-1}{n}\right) v_j$, we can substitute $\left(\frac{n-1}{n}\right) v_j$ for b_j:

$$\begin{aligned} Prob(b_i > b_j \text{ for all } j \neq i) &= Prob\left(b_i > \left(\frac{n-1}{n}\right) v_j \text{ for all } j \neq i\right) \\ &= Prob\left(\left(\frac{n}{n-1}\right) b_i > v_j \text{ for all } j \neq i\right) \\ &= \prod_{j \neq i} Prob\left(\left(\frac{n}{n-1}\right) b_i > v_j\right). \end{aligned}$$

Games of Incomplete Information

First Price Sealed Bid Auction: IPV

- Using the cdf on other player's valuations, we have:

$$Prob\left(\left(\frac{n}{n-1}\right)b_i > v_j\right) = \left(\frac{n}{n-1}\right)b_i.$$

$$Prob\left(\left(\frac{n}{n-1}\right)b_i > v_j \text{ for all } j \neq i\right) = \left[\left(\frac{n}{n-1}\right)b_i\right]^{n-1}.$$

- Bidder i's expected payoff:

$$\pi_i\left(b_i, v_i\right) = \left[\left(\frac{n}{n-1}\right)b_i\right]^{n-1}\left(v_i - b_i\right).$$

Games of Incomplete Information

First Price Sealed Bid Auction: IPV

- First-order condition:

$$\begin{aligned}\frac{\partial \pi_i\left(b_i, v_i\right)}{\partial b_i} &= 0 = -\left[\left(\frac{n}{n-1}\right) b_i\right]^{n-1} \\ &\quad + (n-1)\left(\frac{n}{n-1}\right)\left[\left(\frac{n}{n-1}\right) b_i\right]^{n-2}\left(v_i - b_i\right) \Rightarrow\end{aligned}$$

$$b_i^* = \left(\frac{n-1}{n}\right) v_i.$$

- Each bidder then proportionately shades her bid by $\frac{1}{n}$.
- Since $\frac{n-1}{n}$ is increasing in n, the more bidders there are, the higher is a bidder's bid.

Games of Incomplete Information

First Price Sealed Bid Auction: Common Value

- Suppose there are $n \geq 2$ bidders.
- The true value of the object being auctioned is v and is the same for all bidders.
- Each bidder gets a noisy signal of v which is chosen by Nature from the interval $[0, 1]$ according to a uniform distribution.
- The cdf on bidder i's signal, denoted s_i, is:

$$F(s_i) = \begin{cases} 0 & \text{if } s_i < 0 \\ s_i & \text{if } 0 \leq s_i \leq 1 \\ 1 & \text{if } 1 < s_i. \end{cases}$$

  Signals are independent across bidders.

- The signal of bidder i is known only to him; thus, a bidder's signal is his type and the type space is $[0, 1]$.
- The true value equals the average of all bidders' signals:

$$v = \left(\frac{1}{n}\right) \sum_{j=1}^{n} s_j.$$

Games of Incomplete Information

First Price Sealed Bid Auction: Common Value

- Bidders participate in a first-price sealed bid auction which means that if bidder i wins then his realized payoff is $v - b_i$ where b_i is his bid, though he doesn't learn v until after he's won.
- In deriving a Bayes-Nash equilibrium, conjecture that it is linear in a bidder's signal:
$$b_i = \alpha s_i, \text{ where } \alpha > 0.$$
- Bidder i's expected payoff is the probability that he wins times his expected payoff *conditional on having submitted the highest bid*:
$$Prob\left(b_i > b_j \text{ for all } j \neq i\right) \times \left\{E\left[v \left| s_i, b_i > b_j \text{ for all } j \neq i\right.\right] - b_i\right\}$$

Games of Incomplete Information

First Price Sealed Bid Auction: Common Value

- Use the property that the other bidders are conjectured to use the bidding rule $b_j = \alpha s_j$ and substitute αs_j for b_j in the expected payoff:

$$Prob\left(b_i > \alpha s_j \text{ for all } j \neq i\right) \times \left\{E\left[v \left| s_i, b_i > \alpha s_j \text{ for all } j \neq i\right.\right] - b_i\right\}$$

$$\begin{aligned} = \; & Prob\left(\frac{b_i}{\alpha} > s_j \text{ for all } j \neq i\right) \\ & \times \left\{E\left[\left(\frac{1}{n}\right)\left(s_i + \sum_{j \neq i} s_j\right) \middle| s_i, \frac{b_i}{\alpha} > s_j \text{ for all } j \neq i\right] - b_i\right\} \\ = \; & Prob\left(\frac{b_i}{\alpha} > s_j \text{ for all } j \neq i\right) \\ & \times \left\{\left(\frac{s_i}{n}\right) + \left(\frac{1}{n}\right)\sum_{j \neq i} E\left[s_j \middle| \frac{b_i}{\alpha} > s_j\right] - b_i\right\} \end{aligned}$$

Games of Incomplete Information

First Price Sealed Bid Auction: Common Value

- Since

$$Prob\left(\frac{b_i}{\alpha} > s_j\right) = \frac{b_i}{\alpha}$$

and

$$E\left[s_j \left| \frac{b_i}{\alpha} > s_j \right.\right] = \frac{b_i}{2\alpha}.$$

then bidder i's expected payoff is

$$\left(\frac{b_i}{\alpha}\right)^{n-1}\left[\left(\frac{s_i}{n}\right) + \left(\frac{n-1}{n}\right)\left(\frac{b_i}{2\alpha}\right) - b_i\right]. \tag{5}$$

Games of Incomplete Information

First Price Sealed Bid Auction: Common Value

- First-order condition:

$$\begin{aligned}\frac{\partial \cdot}{\partial b_i} &= 0 = (n-1)\left(\frac{1}{\alpha}\right)\left(\frac{b_i}{\alpha}\right)^{n-2}\left[\left(\frac{s_i}{n}\right)+\left(\frac{n-1}{n}\right)\left(\frac{b_i}{2\alpha}\right)-b_i\right] \\ &\quad +\left(\frac{b_i}{\alpha}\right)^{n-1}\left(\frac{n-1-2\alpha n}{2\alpha n}\right)\end{aligned}$$

- Solving this equation for b_i,

$$b_i = \left(\frac{2\alpha}{2\alpha n-(n-1)}\right)\left(\frac{n-1}{n}\right)s_i. \qquad (6)$$

- Solve for α by equating α with the coefficient on s_i in (6):

$$\alpha = \left(\frac{2\alpha}{2\alpha n-(n-1)}\right)\left(\frac{n-1}{n}\right) \Rightarrow \alpha = \frac{(n+2)(n-1)}{2n^2}.$$

Games of Incomplete Information

First Price Sealed Bid Auction: Common Value

- Symmetric Bayes-Nash equilibrium bidding rule:

$$b_i = \left(\frac{n+2}{2n}\right)\left(\frac{n-1}{n}\right)s_i.$$

- As the number of bidders grows,

$$\lim_{n\to\infty}\left(\frac{n+2}{2n}\right)\left(\frac{n-1}{n}\right)s_i = \frac{s_i}{2}$$

even though

$$\begin{aligned}\lim_{n\to\infty} E\left[v\,|s_i\right] &= \lim_{n\to\infty}\left[\left(\frac{1}{n}\right)s_i + \left(\frac{1}{n}\right)\sum_{j\neq i} E\left[s_j\right]\right] \\ &= \lim_{n\to\infty}\left[\left(\frac{1}{n}\right)s_i + \left(\frac{n-1}{n}\right)\left(\frac{1}{2}\right)\right] = \frac{1}{2}.\end{aligned}$$

Games of Incomplete Information

Second Price Sealed Bid Auction

- Second price sealed bid auction: Bidder with the highest bid wins the item and pays a price equal to the second highest bid.
- A bidder optimally submits a bid *equal* to her valuation, *regardless of how other bidders bid*. There is no shading of the bid below one's valuation as with the first price sealed bid auction.
- Claim: Bidding one's valuation weakly dominates any other bid.
  - A bid of v_i gives at least as high a payoff as b' (when $b' \neq v_i$) for all bids of the other bidders.
  - There are some bids for the other bidders such that bidding v_i gives a strictly higher payoff than bidding b'.
  - Define B_{-i} as the maximum bid of all bidders excluding bidder i:

$$B_{-i} \equiv \max\left\{b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n\right\}.$$

Games of Incomplete Information

Second Price Sealed Bid Auction

- Compare a bid of v_i with a lower bid, denoted $b' < v_i$. Payoffs are:

Case	Condition on B_{-i}	Bidding v_i	Bidding b'	Which is better?
I	$v_i < B_{-i}$	0	0	no difference
II	$b\prime < B_{-i} < v_i$	$v_i - B_{-i} > 0$	0	v_i
III	$B_{-i} < b'$	$v_i - B_{-i}$	$v_i - B_{-i}$	no difference

- Compare a bid of v_i with a higher bid, denoted $b' > v_i$. Payoffs are:

Condition on B_{-i}	Bidding v_i	Bidding b'	Which is better?
$b' < B_{-i}$	0	0	no difference
$v_i < B_{-i} < b'$	0	$v_i - B_{-i} < 0$	v_i
$B_{-i} < v_i$	$v_i - B_{-i}$	$v_i - B_{-i}$	no difference

Games of Incomplete Information

Average Bid Auction

- Procurement auction with ex post screening of bids
  - Consider a state auctioning off road maintenance contracts using a FPSB.
  - A concern is that the firm with the lowest bid may not be reliable and ultimately default.
- Average bid auction
  - Florida Department of Transportation: "If five or more responsive bids are received, the Department will average the bids, excluding the highest and lowest responsive bids. If there or four responsive bids are received, the Department will average all bids. Award of the Contract will be to the winner who submitted the responsive bid closest to the average of those bids."
  - Over 1998-2006 in Italy, a form of the Average Bid Auction was the only mechanism allowed to procure contracts for projects worth less than 5 million Euros.

Games of Incomplete Information

Average Bid Auction

- $n \geq 3$ firms each have its cost randomly drawn from $\left[\underline{c}, \overline{c}\right]$.
- A firm's cost is private information where c_i denote firm i's cost.
- Auction procedure
  - Firms simultaneously choose bids from $[0, R]$ where $R\,(> \overline{c})$ is the reserve price of the local government. Let b_i denote bidder i's bid.
  - The contract is awarded to firm i where

$$\left| b_i - \left(\frac{1}{n}\right) \sum_{j=1}^{n} b_j \right| \leq \left| b_h - \left(\frac{1}{n}\right) \sum_{j=1}^{n} b_j \right|, \ \forall h \neq i.$$

    In the event of ties, the contract is randomly allocated among the firms with the bid closest to the average.
  - If firm i wins the contract then its payoff is $b_i - c_i$.

Games of Incomplete Information

Average Bid Auction

- $b' \in [\overline{c}, R]$ is a symmetric Bayes-Nash equilibrium.
- Assume the other $n-1$ firms bid b'.
- Firm i's payoff from bidding $b_i \neq b'$ is zero as its probability of winning is zero.
  - Difference between firm i's bid and the average bid is
    $$\left| b_i - \left(\frac{1}{n}\right)\left(b_i + (n-1)\, b'\right) \right| = \left(\frac{n-1}{n}\right)\left| b_i - b' \right| .$$
  - Difference between firm h's bid and the average bid, for $h \neq i$, is
    $$\left| b' - \left(\frac{1}{n}\right)\left(b_i + (n-1)\, b'\right) \right| = \left(\frac{1}{n}\right)\left| b_i - b' \right| .$$
  - Since $n \geq 3$ then
    $$\left(\frac{n-1}{n}\right)\left| b_i - b' \right| > \left(\frac{1}{n}\right)\left| b_i - b' \right|$$
    and, therefore, firm i loses the auction for sure.

Games of Incomplete Information

Average Bid Auction

- Firm i's payoff from bidding $b_i = b'$ is

$$\left(\frac{1}{n}\right)\left(b' - c_i\right) \geq 0 \text{ as } c_i \leq \overline{c} \leq b'.$$

- Bidding b' yields a non-negative payoff (and a positive payoff when $c_i < b'$), while bidding different from b' yields a zero payoff.
- Properties of Bayes-Nash equilibria for the Average Bid Auction.
  - Many equilibria with the winning bid unrelated to cost.
  - Ripe for collusion as if firms coordinate on bidding b', there is no incentive to deviate.

Signaling Games

- A signaling game involves two players, the sender and the receiver.
- Stage 1: Nature chooses the sender's type.
- Stage 2: The sender learns her type and chooses an action.
- Stage 3: The receiver observes the sender's action, modifies her beliefs about the sender's type in light of this new information, and chooses an action.

Signaling Games

- In a signaling game, a receiver starts with a set of beliefs about the sender's type (prior beliefs).
    - Priors beliefs are the probabilities assigned by nature
    - After observing the sender's action (signal), the receiver modifies her original beliefs (posterior beliefs).
- Perfect Bayes Nash equilibrium is founded on two key concepts:
    - Sequential Rationality: At each point in a game, a player's strategy prescribes an optimal action, given her beliefs about what other players will do.
    - Consistent Beliefs: A receiver has consistent beliefs if her posterior beliefs are consistent with the sender's acting in her own best interest.

Signaling Games

- A *separating strategy* is a strategy that assigns a distinct action to each type of player. Hence, the receiver can separate out each player's type from her observed play.
- A *pooling strategy* is a strategy where all sender choose the same action.
- Perfect Bayes-Nash equilibrium for a signaling game:
    - For each type of the sender, the sender's strategy prescribes an action that maximizes the sender's expected payoff, given how the receiver will respond.
    - For each action of the sender, the receiver's strategy prescribes an action that maximizes the receiver's expected payoff, given the receiver's posterior beliefs about the sender's type.
    - The receiver's posterior beliefs about the sender's type, conditional on observing the sender's action, are consistent with the sender's strategy and Bayes Rule (when applicable).

Signaling Games

Used Car Market

- Suppose the car for sale can be of three quality levels: low, moderate, or high. The seller knows the true quality but the buyer does not; thus a seller's type is the car's quality.
- The buyer initially believes there is a 20% chance the car is of high quality, 50% it is of moderate quality, and 30% it is of low quality.
- Using these probabilities, Nature determines the seller's type.
- The seller then decides whether to put the car up for sale and, if so, what price to post. If the car is for sale, the buyer observes the price and decides whether or not to buy it.
- A seller's strategy assigns a price to each quality of the car.
- A buyer's strategy tells him whether to accept or decline each possible price (in the event the car is for sale).

Signaling Games

Used Car Market

- Seller's payoff
  - If sold, then the price paid.
  - If not sold, then the value of the car to the seller.
- Buyer's payoff
  - If bought, then the value of the car to the buyer minus the price paid.
  - If not bought, then zero.

Used Car Market

Quality	Probability	Value to seller	Value to buyer
High	.20	20,000	24,000
Moderate	.50	15,000	18,000
Low	.30	10,000	12,000

- With complete information, there is always a basis for a sale because the buyer values the car more than the seller.

Signaling Games

Used Car Market

- *Is there an equilibrium in which higher quality cars sell for more?*
- Consider a separating strategy in which the owner posts a price of P^h for a high quality car, a price of P^m for moderate quality, and P^l for low low quality.
- Consistent buyer beliefs assign prob. one to the car being of high quality when the price is P^h, prob. one to the car being of moderate quality when the price is P^m, and prob. one to the car being of low quality when the price is P^l.
- Buyer is willing to buy
    - at the high price if $P^h \leq 24,000$
    - at the moderate price if $P^m \leq 18,000$
    - at the low price if $P^l \leq 12,000$.
- Seller's strategy is not optimal as the owner of a low quality car prefers to price at P^h than at P^l.
- Conclusion: Higher quality cars cannot sell for more.

Signaling Games

Used Car Market

- *Is there an equilibrium in which all quality type cars are sold and they sell for the same price?*
- Consider a pooling strategy in which the seller sets the same price, $\overline{P}$, regardless of quality.
- Seller's strategy: Price at $\overline{P}$ whether the car is of low, moderate, or high quality.
- Buyer's strategy:
  - If $P \leq \overline{P}$ then buy the car.
  - If $P > \overline{P}$ then do not buy the car.
- Buyer's beliefs: For any price, the car is believed to be of low quality with probability .3, moderate quality with probability .5, and high quality with probability .2.

Signaling Games

Used Car Market

- Buyer's beliefs are consistent.
- Buyer's strategy is optimal iff

$$.2 \times \left(24,000 - \overline{P}\right) + .5 \times \left(18,000 - \overline{P}\right) + .3 \times \left(12,000 - \overline{P}\right) \geq 0$$

$$\Rightarrow \overline{P} \leq 17,400.$$

- Seller's strategy is optimal iff

$$\begin{aligned} \overline{P} &\geq 20,000 \text{ when the car is high quality} \\ \overline{P} &\geq 15,000 \text{ when the car is moderate quality} \\ \overline{P} &\geq 10,000 \text{ when the car is low quality} \end{aligned}$$

which implies $\overline{P} \geq 20,000$.

- Conclusion: All cars cannot sell at the same price.

Signaling Games

Used Car Market

- Consider a semi-pooling strategy for the seller:
- Seller's strategy:
  - If the car is of low or moderate quality then price at $\widetilde{P}$.
  - If the car is of high quality then do not put the car up for sale.
- Buyer's strategy:
  - If $P \leq \widetilde{P}$ then buy the car.
  - If $P > \widetilde{P}$ then do not buy the car.
- Buyer's beliefs: For any price, the car is believed to be of low quality with probability .375, moderate quality with probability .625, and high quality with probability 0.

Signaling Games

Used Car Market

- Seller's strategy is optimal when $15,000 \leq \widetilde{P} \leq 20,000$.
  - A seller with a moderate quality car finds it optimal to sell if and only if $\widetilde{P} \geq 15,000$.
  - If $\widetilde{P} \geq 15,000$ then an owner of a low quality car also finds it optimal to sell.
  - If the car is of high quality then it is optimal to keep the car off of the market iff $\widetilde{P} \leq 20,000$.
- Buyer's beliefs
  - If the price is $\widetilde{P}$ then, according to the seller's strategy, the car must be of low or moderate quality.
  - Consistency requires that probability zero be attached to high quality and, using Bayes Rule, the posterior probability that the car is low quality is $.375\,(= .3/.8)$ and moderate quality is $.625\,(= .5/.8)$.

Signaling Games

Used Car Market

- Given those beliefs, the buyer's strategy of buying at a price of $\widetilde{P}$ is optimal iff:

$$.625 \times \left(18,000 - \widetilde{P}\right) + .375 \times \left(12,000 - \widetilde{P}\right) \geq 0 \Rightarrow \widetilde{P} \leq 15,750.$$

- PBNE iff $15,000 \leq \widetilde{P} \leq 15,750.$

Signaling Games

Suicide

> *Suicidal behavior ... includes a heterogeneous spectrum of suicide attempts that range from highly lethal attempts (in which survival is the result of good fortune) to low-lethality attempts that occur in the context of a social crisis and contain a strong element of an appeal for help. Men tend to use means that are more lethal ... In contrast, women tend to use less lethal means of suicide ... and they more commonly express an appeal for help by conducting the attempt in a manner that favors discovery and rescue. Thus, suicidal behavior has two dimensions. The first dimension is the degree of medical lethality or damage resulting from the suicide attempt. The second dimension relates to suicidal intent and measures the degree of preparation, the desire to die versus the desire to live, and the chances of discovery. [J. Mann, "A Current Perspective of Suicide and Attempted Suicide," Annals of Internal Medicine, 2002.]*

Signaling Games

Suicide

- Stage 1: Nature chooses Marilyn's mental state (type). She is deeply depressed (*depressed*) with probability $1 - p$ and mildly depressed (*normal*) with probability p.
- Stage 2: Marilyn learns her mental state and then chooses a method of suicide summarized by s which is the probability that it causes death; $0 \leq s \leq 1$.
- Stage 3: Given the method of suicide chosen, Nature determines whether Marilyn survives. If s was chosen then she dies with probability s.
- Stage 4: Sigmund observes the suicide method, s, and, if Marilyn survives, he decides whether to offer treatment.

Signaling Games

Suicide

- A strategy for Marilyn maps from {depressed, normal} to $[0, 1]$, a method of suicide s.
- A strategy for Sigmund maps from $[0, 1]$ (an observed method of suicide) into {treatment, no treatment}.
- Sigmund's payoffs are such that he prefers to offer treatment only when she is depressed.

Sigmund's Payoffs

Marilyn's type	Treatment	No treatment
Normal	1	2
Depressed	3	-1

Signaling Games

Suicide

- Marilyn's payoffs are always higher when Sigmund provides treatment.
  - If normal, Marilyn prefers life to death.
  - If depressed, Marilyn prefers life with treatment to death but prefers death to life without treatment.

Marilyn's Payoffs

Marilyn's type	Sigmund treats	Sigmund does not treat	Death
Normal	3	1	0
Depressed	4	-1	0

Signaling Games

Suicide

- Separating perfect Bayes-Nash equilibrium in which a suicide attempt brings forth treatment by Sigmund.
- Marilyn's strategy:
  - If normal then $s = 0$.
  - If depressed then $s = s^*$.
- Sigmund's strategy:
  - If $s < s^*$ then do not offer treatment.
  - If $s \geq s^*$ then offer treatment.
- Sigmund's beliefs:
  - If $s < s^*$ then Marilyn is normal with probability one.
  - If $s \geq s^*$ then Marilyn is depressed with probability one.

Signaling Games

Suicide

- Consistency of Sigmund's beliefs
  - If $s = 0$ then prob. $= 1$ that Marilyn is normal since only a normal type chooses $s = 0$.
  - If $s = s^*$ then prob. $= 1$ that Marilyn is depressed since only a depressed type chooses $s = s^*$.
  - If $s \notin \{0, s^*\}$ then any beliefs are consistent.
- Optimality of Sigmund's strategy.
  - If he observes $s < s^*$ then he believes she is normal in which case he prefers no treatment.
  - If he observes $s \geq s^*$ then he believes she is depressed in which case he prefers treatment.

Signaling Games

Suicide

- Optimality of Marilyn's strategy when she is normal.
  - Marilyn would never choose a suicide method between 0 and s^* since doing so risks death but fails to induce treatment. Hence, no suicide attempt $(s = 0)$ is superior to a method less lethal than s^*.
  - If she chooses a suicide method lethal enough to bring forth treatment $(s \geq s^*)$ then, if she survives, her expected payoff is

    $$s \times 0 + (1 - s) \times 3 = 3\,(1 - s)\,,$$

    which is maximized at $s = s^*$. If s^* is sufficient to induce treatment then there is no purpose in choosing a more lethal method.
  - She prefers $s = 0$ to $s = s^*$ iff

    $$1 \geq 3\,(1 - s^*) \Leftrightarrow s^* \geq 2/3.$$

Signaling Games

Suicide

- Optimality of Marilyn's strategy when she is depressed.
  - s^* is superior to a more lethal method since her expected payoff from s, when $s \geq s^*$, is
    $$s \times 0 + (1 - s) \times 4 = 4(1 - s).$$
  - If Marilyn chooses $s < s^*$ then she does not receive treatment so her expected payoff is
    $$s \times 0 + (1 - s) \times (-1) = s - 1.$$
  - s^* is optimal because
    $$4(1 - s^*) \geq s - 1 \text{ for all } s < s^* (< 1).$$
- It is then a perfect Bayes-Nash equilibrium for only a deeply depressed Marilyn to attempt suicide and for Sigmund to offer treatment as long as the method is sufficiently lethal (more specifically, it causes death with probability of at least 2/3).

Cheap Talk Games

It is hard to believe that a man is telling the truth when you know that you would lie if you were in his place. - H. L. Mencken

- A *cheap talk game* is a signaling game where there are no costs to signaling.
- Such a costless action is referred to as a message.
- Costliness of messages - For example, suppose you are considering buying a used car from a seller.
  - If the seller tells you that the car is of good quality, that is cheap talk, since a seller can make that statement regardless of the car's true quality.
  - If the seller offers a warranty, however, the cost of that is higher when the quality is lower, so the signal's cost varies with the type.

Cheap Talk Games

Stock Recommendations

- Are stock recommendations informative?

Stock Recommendations

Recommendation	Frequency	Cumulative Percentage
Strong Buy	38	15.2%
Buy	128	66.4%
Hold	70	94.4%
Sell	14	100.0%
Strong Sell	0	100.0%

Source: Dugar and Nathan (1995)

Cheap Talk Games

Stock Recommendations

- Stage 1: Nature determines whether the security analyst believes the stock will *outperform*, move with the market (*neutral*), or *underperform*. Each occurs with prob. 1/3.
- Stage 2: An analyst learns her type and chooses a recommendation: *buy*, *hold*, or *sell*.
- Stage 3: An investor learns the analyst's recommendation - though doesn't know what the analyst truly believes - and decides whether to *buy*, *hold*, or *sell*.

Cheap Talk Games

Stock Recommendations

- Investor's payoff equals
  - 1 from pursuing the best action (under full information) which means buying when the stock is predicted to outperform, holding when it is to move with the market, and selling when it is to underperform
  - -1 from choosing the least desirable action which means selling when the stock is to outperform and buying when it is to underperform
  - 0 otherwise.
- Analyst's payoff equals the investor's payoff plus a when the investor buys and less b when she sells, where $a, b > 0$.
  - It is increasing in the investor's payoff because her compensation is higher when her client's portfolio performs better.
  - Adding a or subtracting b is motivated by investment banking considerations. The analyst (and her company) are harmed when clients are induced to sell a stock, and benefit when they are induced to buy.

Cheap Talk Games

Stock Recommendations

State	Action	Analyst's payoff	Investor's payoff
Outperform	Buy	$a+1$	1
Outperform	Hold	0	0
Outperform	Sell	$-b-1$	-1
Neutral	Buy	a	0
Neutral	Hold	1	1
Neutral	Sell	$-b$	0
Underperform	Buy	$a-1$	-1
Underperform	Hold	0	0
Underperform	Sell	$1-b$	1

Cheap Talk Games

Stock Recommendations: Separating Equilibrium

- Analyst's strategy:
    - Recommend buy when the stock is an outperform.
    - Recommend hold when the stock is a neutral.
    - Recommend sell when the stock is an underperform.
- Investor's strategy: Follow the analyst's recommendation.
- Investor's beliefs:
    - When the analyst recommends buy then the stock is an outperform with prob. one.
    - When the analyst recommends hold then the stock is a neutral with prob. one.
    - When the analyst recommends sell then the stock is an underperform with prob. one.
- Consistency of investor's beliefs is obvious.
- Optimality of the investor's strategy, given her beliefs, is obvious.

Cheap Talk Games

Stock Recommendations: Separating Equilibrium

- Optimality of the analyst's strategy
- Suppose the analyst believes the stock will outperform.
  - Hold (sell) recommendation yields a payoff of 0 ($-b-1$).
  - **Buy** recommendation yields a payoff of $a+1$.
- Suppose the analyst believes the stock will perform at the market (neutral).
  - Buy (sell) recommendation yields a payoff of a ($-b$).
  - **Hold** recommendation yields a payoff of 1; which is optimal iff $a \leq 1$.
- Suppose the analyst believes the stock will underperform.
  - Buy (hold) recommendation yields a payoff of $a-1$ (0).
  - **Sell** recommendation yields a payoff of $1-b$; which is optimal iff $1-b \geq \max\{0, a-1\} \Rightarrow 1 \geq b$ and $2 \geq a+b$.
- Equilibrium requires: $a \leq 1,\ b \leq 1,\ a+b \leq 2 \Rightarrow a \leq 1$ and $b \leq 1$.

Cheap Talk Games

Stock Recommendations: Semi-Separating Equilibrium

- Suppose $b > 1$ so that it is highly detrimental to induce clients to sell. It is then no longer an equilibrium for an analyst to always reveal the truth.
- Consider recommendations being partially informative.
- Analyst's strategy:
  - Recommend buy when the stock is an outperform or neutral.
  - Recommend hold when the stock is an underperform.
- Investor's strategy
  - Buy when the analyst recommends buy.
  - Sell when the analyst recommends hold or sell.
- Investor's beliefs:
  - When the analyst recommends buy, assign probability 1/2 to outperform and 1/2 to neutral.
  - When the analyst recommends hold, assign prob. one to underperform.
  - When the analyst recommends sell, assign prob. one to underperform.

Cheap Talk Games

Stock Recommendations: Semi-Separating Equilibrium

- Consistency of investor's beliefs.
- Analyst recommends buy.
  - Analyst does so when the stock is either an outperform or a neutral.
  - Investor should assign probability 0 to the stock being an underperform.
  - By Bayes Rule, should assign probability 1/2 to being an outperform.
- Analyst recommends hold.
  - Analyst does so when the stock is an underperform.
  - Investor should assign prob. 1 to the stock being an underperform.
- Analyst recommends sell - consistency places no restrictions on beliefs.

Cheap Talk Games

Stock Recommendations: Semi-Separating Equilibrium

- Optimality of the investor's strategy.
- Analyst recommends buy.
  - Expected payoff from buying is
    $$\frac{1}{2} \times 1 + \frac{1}{2} \times 0 = \frac{1}{2}.$$
  - Expected payoff from holding is
    $$\frac{1}{2} \times 0 + \frac{1}{2} \times 1 = \frac{1}{2}.$$
  - Expected payoff from selling is
    $$\frac{1}{2} \times 0 + \frac{1}{2} \times (-1) = -\frac{1}{2}.$$
  - Buying the stock is indeed optimal.
- Analyst recommends hold or sell.
  - Investor believes the stock is an underperform and her strategy appropriately calls for her to sell.

Cheap Talk Games

Stock Recommendations: Semi-Separating Equilibrium

- Optimality of the analyst's strategy.
- Stock is an outperform (and strategy calls for "buy recommendation")
    - Payoff is $a+1$ from a buy recommendation, $-b-1$ from a hold or a sell recommendation.
- Stock is a neutral (and strategy calls for "buy recommendation")
    - Payoff is a from a buy recommendation, $-b$ from a hold or a sell recommendation.
- Stock is an underperform (and strategy calls for "hold recommendation")
    - Payoff is $a-1$ from a buy recommendation, $1-b$ from a hold or a sell recommendation
    - Optimality requires $1-b \geq a-1 \Rightarrow a+b \leq 2$.
- Equilibrium condition: $a+b \leq 2$.

Cheap Talk Games

Stock Recommendations

- Equilibrium conditions
  - Separating equilibrium (fully informative recommendations) exists when: $a \leq 1$ and $b \leq 1$.
  - Semi-separating equilibrium (partially informative recommendations) exists when: $a + b \leq 2$.
- When b (or a) is higher, the interests of the analyst and investor diverge to a greater degree as the former is more concerned about the impact of purchases and sales of a company's stock on its investment banking business. This makes it more difficult for the analyst to provide truthful recommendations to his investors.
- General lesson: Messages are less informative, the greater the conflict of interests between the sender and receiver.

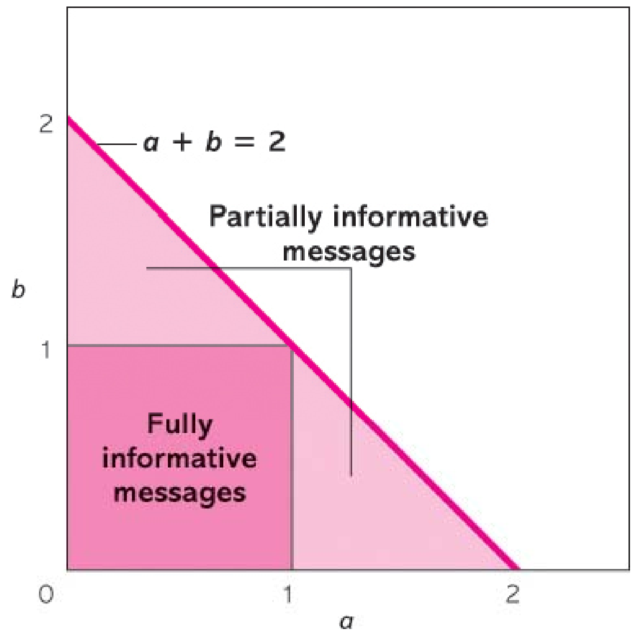


Figure 12.4 The Information Content of Stock Recommendations
Harrington: Games, Strategies, and Decision Making, First Edition

Cheap Talk Games

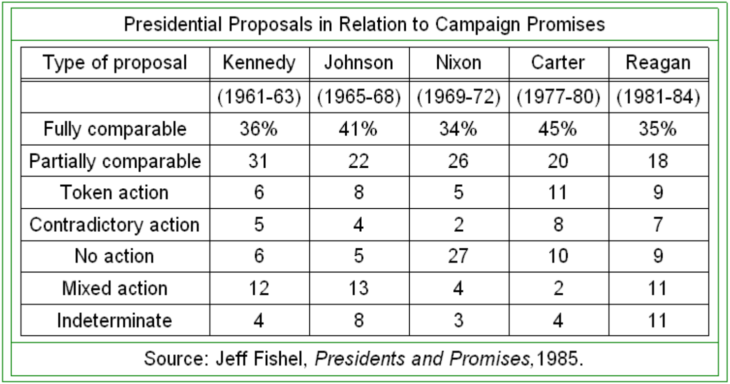
Campaign Promises

A politician will always tip off his true belief by stating the opposite at the beginning of the sentence. For maximum comprehension, do not start listening until the first clause is concluded. Begin instead at the word "but" which begins the second, or active, clause. This is the way to tell a liberal from a conservative – before they tell you. Thus: "I have always believed in a strong national defense, second to none, but ... " (a liberal, about to propose a $20 billion defense cut). - Frank Mankiewicz

- When are campaign promises to be believed?
  - When running for office in 1988, George H. W. Bush made an unambiguous pledge: "Read my lips ... no new taxes!"
  - Gallup poll: 68% of those surveyed believed he would *raise* taxes if elected. In fact, he did.

Cheap Talk Games

Campaign Promises



Presidential Proposals in Relation to Campaign Promises

Type of proposal	Kennedy	Johnson	Nixon	Carter	Reagan
	(1961-63)	(1965-68)	(1969-72)	(1977-80)	(1981-84)
Fully comparable	36%	41%	34%	45%	35%
Partially comparable	31	22	26	20	18
Token action	6	8	5	11	9
Contradictory action	5	4	2	8	7
No action	6	5	27	10	9
Mixed action	12	13	4	2	11
Indeterminate	4	8	3	4	11

Source: Jeff Fishel, *Presidents and Promises*, 1985.

Cheap Talk Games

Campaign Promises

- Candidates 1 and 2 compete for office.
- A candidate's ideology is private information and is either conservative or liberal.
- Nature chooses a candidate to be a liberal with probability L and a conservative with probability $1 - L$.
- The electorate's preferences across conservative and liberal policies are private information to them as well. With probability l, the median voter is liberal, and with probability $1 - l$ is conservative.
- Stage 1: Nature chooses the types of candidate 1, candidate 2, and the (median) voter.
- Stage 2: After learning their types, the two candidates simultaneously announce a policy.
- Stage 3: After learning her type and observing the candidates' announced policies, the voter updates her beliefs as to their ideologies and then votes.

Cheap Talk Games

Campaign Promises

- Candidate payoffs
  - A candidate cares about being elected and, in that event, what policy is implemented.
  - A conservative (liberal) elected official would like to see a conservative (liberal) policy implemented.
  - A candidate's payoff is zero if not elected.
  - If he is elected and
    - ★ his less preferred policy is implemented then his payoff is 1.
    - ★ his more preferred policy is implemented then his payoff is $u \geq 1$.
  - If $u = 1$ then he is purely office-motivated.
- Voter's payoff
  - from having her more preferred policy implemented is 1
  - from having her less preferred policy implemented is 0.

Cheap Talk Games

Campaign Promises

- Determination of policy
- Post-election policy depends on the ideologies of the winning candidate and the electorate.
- If the winning candidate and the median voter have the same ideal policy then that policy is implemented for sure.
- If they have different preferences then the winning candidate's preferred policy is implemented with probability *h*.
    - *h* measures the power of the office.

Cheap Talk Games

Campaign Promises: Separating Equilibrium

- Symmetric candidate strategy:
  - announce a liberal policy if liberal.
  - announce a conservative policy if conservative.
- Voter's strategy:
  - If both candidates announce the same policy then randomize with equal probability.
  - If liberal and candidates announce different policies then vote for the one who supports a liberal policy.
  - If conservative and candidates announce different policies then vote for the one who supports a conservative policy.
- Voter's beliefs:
  - If a candidate announces a liberal policy then assign probability one to him being liberal.
  - If a candidate announces a conservative policy then assign probability one to him being conservative.

Cheap Talk Games

Campaign Promises: Separating Equilibrium

- Consistency of voter's beliefs and optimality of voter's strategy are obvious.
- Optimality of candidate 1's strategy when he is a liberal.

		Expected Payoff	
Cand. 2	Voter	Liberal Platform	Conservative Platform
Liberal	Liberal	$(1/2)\,u$	0
Cons.	Liberal	u	$(1/2)\,u$
Liberal	Cons.	$(1/2)\,(hu+(1-h))$	$hu+(1-h)$
Cons.	Cons.	0	$(1/2)\,(hu+(1-h))$

- $u\,(\geq 1)$ is the payoff when elected and more preferred policy is implemented.
- 1 is the payoff when elected and less preferred policy is implemented.
- h is the prob. the winning candidate's policy is implemented when it differs from the electorate's.

Cheap Talk Games

Campaign Promises: Separating Equilibrium

- Assume $L = 1/2$ and $h = 1/2$.
- A liberal platform is optimal when candidate 1 is liberal iff

$$l \geq \frac{u+1}{3u+1}.$$

- Probability that the median voter is liberal, l, must be sufficiently great.

Cheap Talk Games

Campaign Promises: Separating Equilibrium

- A conservative platform is optimal when candidate 1 is conservative iff

$$\frac{2u}{3u+1} \geq l.$$

- Probability that the median voter is liberal, l, must be sufficiently small.
- Separating equilibrium exists iff

$$\frac{2u}{3u+1} \geq l \geq \frac{u+1}{3u+1}.$$

- If $u = 1$ (so the candidates are pure office-seekers) then it only holds for the very special case of $l = 1/2$. Campaign promises ought to be ignored if candidates only care about holding office.

Cheap Talk Games

Campaign Promises: Separating Equilibrium

- Separating equilibrium exists iff

$$\frac{2u}{3u+1} \geq I \geq \frac{u+1}{3u+1}.$$

- As u increases, the interval $\left[\frac{u+1}{3u+1}, \frac{2u}{3u+1}\right]$ expands and it is easier for messages to be informative.
  - As u rises, it becomes more important for a candidate to be elected *when* voters have the same ideology as then he'll be able to implement his ideal policy.
  - Misleading the electorate to get into office may increase the chances of being elected but lowers the chances of being able to implement one's preferred policy.

Repeated Games

Cooperation and Price Fixing

> *The competitor is our friend and the customer is our enemy. - Unofficial motto of Archer Daniels Midland*

- Cartel composed of fine arts auction houses, Christie's and Sotheby's.
- Cartel duration: April 1993 to February 2000.

Sotheby's: Net Income (Red), Revenue (Blue), 1987-2005

Repeated Games

Cooperation and Price Fixing

- "Price" is the commission rate or the percentage of the final bid price.

Commission Rates - 1995 (Christie's)

Annual Sales	Commission
Up to \$99,999	10%
\$100,000-\$249,999	9%
\$250,000-\$499,999	8%
\$500,000-\$999,999	6%
\$1,000,000-\$2,499,999	5%
\$2,500,000-\$4,999,999	4%
\$5,000,000 and above	2%

- Assume possible (constant) commission rates: 2%, 4%, 6%, 8%.

Repeated Games

Cooperation and Price Fixing

		Sotheby's			
		2%	4%	6%	8%
Christie's	2%	-20,-20	60,0	140,-60	220,-200
	4%	0,60	100,100	220,60	140,-60
	6%	-60,140	60,220	180,180	320,80
	8%	-200,220	-60,300	80,320	230,230

Repeated Games

Cooperation and Price Fixing

		Sotheby's			
		2%	4%	6%	8%
Christie's	2%	-20,-20	60,0	140,-60	220,-200
	4%	0,60	100,100	220,60	140,-60
	6%	-60,140	60,220	180,180	320,80
	8%	-200,220	-60,300	80,320	230,230

Repeated Games

Cooperation and Price Fixing

- Nash Equilibrium: Both houses charge 4%.

		Sotheby's			
		2%	4%	6%	8%
Christie's	2%	-20,-20	60,0	140,-60	220,-200
	4%	0,60	100,100	220,60	140,-60
	6%	-60,140	60,220	180,180	320,80
	8%	-200,220	-60,300	80,320	230,230

- Basis for collusion
  - Both auction houses are better off with a commission rate of 6% or 8%.
  - They can agree that it is mutually beneficial to raise rates. But is such an agreement self-enforcing?

Repeated Games

Cooperation and Price Fixing

- Infinitely repeated game - The stage game is played infinitely (or indefinitely) often and the history of play is common knowledge
- Strategy set is an infinite sequence of action functions (one for each period) where a period t action function, $x_i^t\left(\cdot\right)$, maps from past commission rates into a commission rate for the current period; $x_i^t : \{2, 4, 6, 8\}^{2(t-1)} \rightarrow \{2, 4, 6, 8\}$.
- Payoff to a player is the present value of its profit stream. If π_i^t is auction house i's profit in period t then its payoff is

$$\pi_i^1 + \delta\pi_i^2 + \delta^2\pi_i^2 + \cdots = \sum_{t=1}^{\infty} \delta^{t-1}\pi_i^t$$

where $0 < \delta < 1$ is a discount factor common to both firms. A constant profit stream π' has a present value of $\pi' / \left(1 - \delta\right)$.

Repeated Games

Cooperation and Price Fixing

- A strategy profile is a *subgame perfect equilibrium for a repeated game* if and only if, in each period and for each history, the prescribed action is optimal for a player given:
    - the other players act according to their strategies in the current period
    - all players (including the player under consideration) act according to their strategies in all future periods.
- A strategy for sustaining a collusive commission rate of 8%.
    - In the initial period, charge 8%.
    - In any future period,
        - charge 8%, if both auction houses charged 8% in all previous periods.
        - charge 4%, otherwise.

Repeated Games

Cooperation and Price Fixing

- SPE Conditions
- Consider period 1 or period $t \geq 2$ when 8% has been charged in all past periods.
    - Strategy prescribes 8% which yields a payoff of

$$230 + \delta \times 230 + \delta^2 \times 230 + \cdots = \frac{230}{1-\delta}.$$

    - The best alternative action is to charge 6% and earn

$$320 + \delta \times 100 + \delta^2 \times 100 + \cdots = 320 + \delta \left(\frac{100}{1-\delta} \right)$$

    - Optimality requires:

$$\frac{230}{1-\delta} \geq 320 + \delta \left(\frac{100}{1-\delta} \right) \Rightarrow \delta \geq 9/22.$$

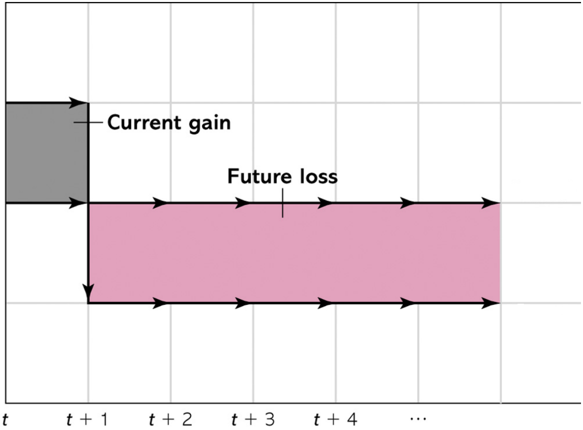
Repeated Games

Cooperation and Price Fixing

- Consider period $t \geq 2$ when 8% has not been charged in some past period.
  - Strategy prescribes 4% which yields a payoff of
    $$\frac{100}{1-\delta}.$$
  - Any other action lowers current profit without changing the future profit stream.
- Strategy profile is a SPE iff $\delta \geq 9/22$.

Repeated Games

Cooperation and Price Fixing



Repeated Games

Cooperation and Price Fixing

- Suppose $\delta < 9/22$ so the previous strategy profile cannot sustain an 8% rate.
- Consider instead supporting a 6% rate.
- SPE condition
  - Strategy prescribes 6% which yields a payoff of
$$\frac{180}{1-\delta}.$$
  - The best alternative action is to charge 4% and earn
$$220+\delta\left(\frac{100}{1-\delta}\right).$$
  - Optimality requires:
$$\frac{180}{1-\delta} \geq 220+\delta\left(\frac{100}{1-\delta}\right) \Rightarrow \delta \geq 1/3.$$

Repeated Games

Cooperation and Price Fixing

- More impatience reduces the ability to collude.
  - If $1/3 \leq \delta < 9/22$ then, with this reward-punishment scheme, the commission rate can be raised from 4% to 6% though not as high as 8%.
  - An auction house is more tempted to cheat and earn higher short-run profit. This is offset by reducing the collusive rate.

Repeated Games

Cooperation and Price Fixing

- Consider a weaker punishment of one-period reversion to the stage game NE of (4%, 4%).
- Strategy
  - In the initial period, each auction house charges 8%.
  - In any future period,
    - each charges 8%, if both auction houses charged 4% or both charged 8% in the previous period.
    - each charges 4%, otherwise.

Repeated Games

Cooperation and Price Fixing

- Consider period 1 or period $t \geq 2$ when both houses charged 4% or charged 8% in the preceding period.
  - Strategy prescribes 8% which yields a payoff of
$$\frac{230}{1-\delta}.$$
  - The best alternative action is to charge 6% and earn
$$320 + \delta \times 100 + \delta^2 \times \left(\frac{230}{1-\delta}\right)$$
  - Optimality requires:
$$\begin{aligned} \frac{230}{1-\delta} &\geq 320 + 100\delta + \delta^2 \left(\frac{230}{1-\delta}\right) \Rightarrow \\ 230 + 230\delta &\geq 320 + 100\delta \Rightarrow \delta \geq 9/13. \end{aligned}$$

Repeated Games

Cooperation and Price Fixing

- Consider period $t \geq 2$ when both houses did not charge 4% or 8% in the preceding period.
  - Strategy prescribes 4% which yields a payoff of
  $$100 + \delta \left(\frac{230}{1-\delta} \right).$$
  - Charging 8% yields
  $$-60 + \delta \times 100 + \delta^2 \times \left(\frac{230}{1-\delta} \right),$$
  which is lower.
  - Charging 2% or 6% yields
  $$60 + \delta \times 100 + \delta^2 \times \left(\frac{230}{1-\delta} \right),$$
  which again is lower.
- To support collusion with a weaker punishment requires that firms are more patient.

Repeated Games

Cooperation and Price Fixing

- One-period punishment: Revert to (2%, 2%) - price war more intense than non-collusive outcome.
- Strategy
  - In the initial period, each auction house charges 8%.
  - In any future period,
    - each charges 8%, f both auction houses charged 2% or both charged 8% in the previous period.
    - each charges 2%, otherwise.

Repeated Games

Cooperation and Price Fixing

- Consider period 1 or period $t \geq 2$ when both houses charged 2% or charged 8% in the preceding period.
  - Strategy prescribes 8% which yields a payoff of
  $$\frac{230}{1-\delta}.$$
  - The best alternative action is to charge 6% and earn
  $$320 + \delta \times (-20) + \delta^2 \times \left(\frac{230}{1-\delta}\right)$$
  - Optimality requires:
  $$\begin{aligned} \frac{230}{1-\delta} &\geq 320 - 20\delta + \delta^2 \left(\frac{230}{1-\delta}\right) \Rightarrow \\ 230 + 230\delta &\geq 320 - 20\delta \Rightarrow \delta \geq 9/25. \end{aligned}$$

Repeated Games

Cooperation and Price Fixing

- Do firms want to go through with a price war?
- Consider period $t \geq 2$ when both houses did not charge 2% or 8% in the preceding period.
  - Collusive rate of 2% yields a value to an auction house of

$$-20 + \delta \times \left(\frac{230}{1-\delta} \right)$$

  - Best alternative action is to charge 4% and earn

$$0 + \delta \times (-20) + \delta^2 \times \left(\frac{230}{1-\delta} \right)$$

  - Optimality requires:

$$\begin{aligned} -20 + \delta \times \left(\frac{230}{1-\delta} \right) &\geq \delta \times (-20) + \delta^2 \times \left(\frac{230}{1-\delta} \right) \Rightarrow \\ -20 + 230\delta &\geq -20\delta \Rightarrow \delta \geq 2/25. \end{aligned}$$

Repeated Games

Cooperation and Price Fixing

- Strategy profile is a SPE iff $\delta \geq \max \{9/25, 2/25\} \Leftrightarrow \delta \geq 9/25$.
- A one-period price war (which supports collusion if $\delta \geq 9/25$) is a more severe punishment than permanently reverting to competition (which supports collusion if $\delta \geq 9/22$).

Repeated Games

Cooperative Norms and Institutions

- Consider a large population of infinitely-lived agents who, in each period, are randomly matched into pairs to possibly transact.
- If one or both choose not to engage then each has a zero payoff.
- If they choose to transact, they simultaneously decide whether to engage in honest or dishonest behavior (Prisoners' Dilemma).

Trader 2

		Honest	Dishonest
Trader 1	Honest	1,1	-1,2
	Dishonest	2,-1	0,0

- Given the population is large, the probability that two agents who are transacting today will interact in the future is small. Essentially, interactions are one-shot and each person faces an endless stream of these one-shot situations.

Repeated Games

Cooperative Norms and Institutions

- How is cooperation sustained?
- Community enforcement
  - Punishment for misbehavior by agent j against agent i must be inflicted by agents other than i who encounter agent j in the future.
  - Ostracism - Shun those agents who have misbehaved in the past.
- Monitoring
  - Misbehavior by agent j against agent i must be learned by agent h if agent h is to shun agent j.
  - Gossip - If this is cheap talk, is it credible? What are the incentives to spread the truth rather than false rumors?

Repeated Games

Cooperative Norms and Institutions

- Need to share information about encounters is well-recognized by eBay.

Feedback on eBay (percentage)

Type of Comment	Buyer of Seller	Seller of Buyer
Positive	51.2%	59.5%
Neutral	0.2%	0.2%
Negative	0.3%	1.0%
No comment	48.3%	39.4%
% of comments positive	99.0%	98.0%

- How informative are these messages? Is there a bias against negative feedback?

Repeated Games

Medieval Law Merchant

- In parts of 13th century Europe, the Law Merchant governed many commercial transactions.
- The Law Merchant was a court to which a wronged party could turn for retribution.
- The Law Merchant could pass judgment and award damages but had no enforcement powers to ensure that damages would be paid.
- What made the Law Merchant effective as an institution for promoting honest trading?

Repeated Games

Medieval Law Merchant

- A trader is infinitely-lived and faces a sequence of one-time encounters.
- When two agents meet, each knows nothing of the history of the other agent.
- If one or both traders decide not to participate in a transaction then each receives a zero payoff.
- If they both decide to transact then they play:

		Trader 2	
		Honest	Dishonest
Trader 1	Honest	1,1	-1,2
	Dishonest	2,-1	0,0

- After the encounter is completed, each receives a payoff and never meet again.
- A trader's payoff is the present value of his utility stream with discount factor δ.

Repeated Games

Medieval Law Merchant

- Stage 1 (Query): A trader decides whether to query the LM about any unpaid judgements against the other trader; the price is p.
- Stage 2 (Transaction): The two traders decide whether to transact (i.e., play the trading game).
- Stage 3 (Appeal): Given the outcome of the transaction, a trader decides whether to go the LM; the price is q.
- Stage 4 (Judgment): If either trader went to the LM and one trader behaved dishonestly and the other behaved honestly then the LM awards damages d to the latter. For any other outcome, no damages are awarded.
- Stage 5 (Payment): If a trader was told to pay damages by the LM then he decides whether to do so.
- Stage 6 (Recording): If damages were awarded and were not paid by the guilty party then the unpaid judgement is recorded by the LM.

Repeated Games

Medieval Law Merchant

- Stage 1 (Query):
  - If a trader has no unpaid judgments then he *queries* the LM.
  - If a trader has an unpaid judgment then he *does not query* the LM.
- Stage 2 (Transaction):
  - If either trader failed to query the LM or if a query established that at least one trader has an unpaid judgement, he *does not transact*.
  - If both traders queried the LM and both have no unpaid judgements, then he *transacts* and plays *honestly*.
- Stage 3 (Appeal): If the traders transacted and one trader acted honestly and the other dishonestly then the victim *brings the case* before the LM. Otherwise, he *does not bring the case*.
- Stage 5 (Payment): If a case is brought before the LM and he awards damages then the defendant *pays damages* to the victim if the defendant has no previously unpaid judgment. Otherwise, he *does not pay damages.*

Repeated Games

Medieval Law Merchant

- If this strategy is used by all traders then, assuming traders start with no unpaid judgments, what will happen is:
    - Each trader will go to the Law Merchant to learn about the other trader.
    - Each trader will learn that the other has no unpaid judgments and then they'll honestly transact.
    - Neither will bring a case before the Law Merchant.
    - Each trader entered the period with no unpaid judgments and will exit it with no unpaid judgments.
    - Payoff is $1 - p$ as they get 1 from a honest transaction and paid p to the Law Merchant to learn about the other trader.

Repeated Games

Medieval Law Merchant

- Verifying it is a SPE (when a trader has no unpaid judgments).
- When is it optimal to query the Law Merchant?
  - Future payoff is independent of whether or not he queries the Law Merchant.
    - ★ If he has no unpaid judgments then he expects to leave the period with a clean record.
    - ★ If he has an unpaid judgment then his record is permanently marred.
  - Impact on current payoff
    - ★ If he queries the LM then he expects to learn that the other trader has a clean record in which case they'll transact honestly and earn a current payoff of $1 - p$.
    - ★ If he does not query the LM then there will not be a transaction so his current payoff is zero.
    - ★ It is optimal to query when $p \leq 1$.

Repeated Games

Medieval Law Merchant

- When is it optimal to transact honestly?
  - Suppose both queried the LM and both have no unpaid judgments so they plan to transact.
  - Payoff from acting honestly is
    $$1 + \frac{\delta \left(1 - p\right)}{\left(1 - \delta\right)}.$$
  - Payoff from acting dishonestly is
    $$2 - d + \frac{\delta \left(1 - p\right)}{\left(1 - \delta\right)}.$$
  - It is optimal to act honestly when
    $$1 + \frac{\delta \left(1 - p\right)}{\left(1 - \delta\right)} \geq 2 - d + \frac{\delta \left(1 - p\right)}{\left(1 - \delta\right)} \Leftrightarrow d \geq 1.$$

Repeated Games

Medieval Law Merchant

- When is it optimal for a victim to appeal?
  - Suppose one trader cheated the other.
  - The wronged trader's future payoff is independent of whether he appeals to the LM.
  - If the wronged trader appeals to the LM, his current payoff is $d - q$.
  - If he does not appeal, his current payoff is 0.
  - Appealing is optimal when $d \geq q$.

Repeated Games

Medieval Law Merchant

- When is it optimal to pay damages?
  - Suppose a judgment is made against a trader (who currently has no unpaid judgments).
  - Paying damages results in a payoff of

$$-d+\frac{\delta\left(1-p\right)}{\left(1-\delta\right)}.$$

  - Not paying damages results in a payoff of 0.
  - It is optimal to pay damages when

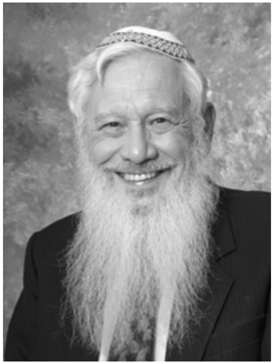
$$\frac{\delta\left(1-p\right)}{\left(1-\delta\right)}\geq d\ .$$

Repeated Games

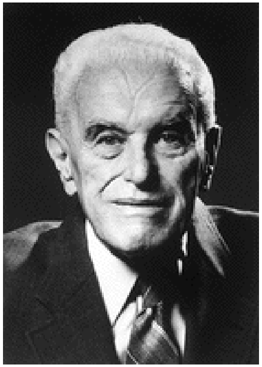
Medieval Law Merchant

1.	Optimal to query the Law Merchant	$p \leq 1$
2.	Optimal to transact honestly	$d \geq 1$
3.	Optimal to appeal to the Law Merchant	$d \geq q$
4.	Optimal to pay damages	$d \leq \frac{\delta(1-p)}{(1-\delta)}$

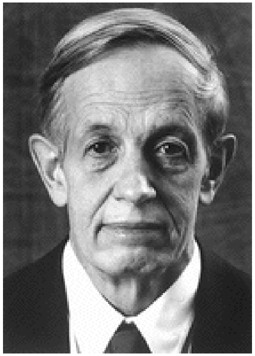
1. The price of a query does not exceed the value of a honest transaction.
2. Damages are sufficiently high so a trader is deterred from cheating.
3. Price of using the Law Merchant is not so high that it is unprofitable to do so when a trader has been victimized.
4. Damages are sufficiently low so that a trader is willing to pay those damages in order to maintain a good reputation through the Law Merchant.



Robert Aumann



John Harsanyi

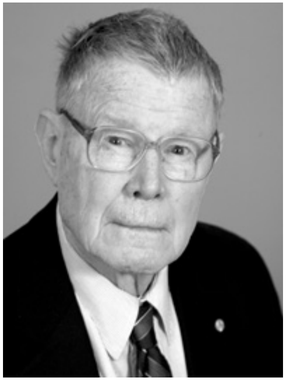


John Nash

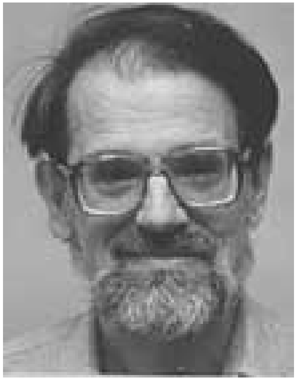
John von Neumann

Roger Myerson

Reinhard Selten



Thomas Schelling



Lloyd Shapley