Appendices A & B
"When Can We Expect a Corporate Leniency Program to Result in Fewer Cartels?,"
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1 Appendix A: Proofs

In reading the proofs, it is useful to have this summary of the solution algorithm for deriving an equilibrium.

1. Given σ and Y and for each η, solve for the maximum market condition (or threshold) for which the ICC is satisfied, $\phi(Y, \sigma, \eta)$.

2. Given σ and for each η, solve for the equilibrium collusive value $Y^*(\sigma, \eta)$ which is a solution to the fixed point problem: $Y^* = \psi(Y^*, \sigma, \eta)$. If there are multiple fixed points, select the maximum. Given $Y^*(\sigma, \eta)$, define the equilibrium threshold $\phi^*(\sigma, \eta)$.

3. Given σ and $\phi^*(\sigma, \eta)$, derive the stationary proportion of type-η industries that are cartels, $C(\sigma, \eta)$, and integrate over industry types to derive the stationary cartel rate:
$$C(\sigma) = \int_{\underline{\eta}}^{\overline{\eta}} C(\sigma, \eta) g(\eta) d\eta = \int_{\underline{\eta}}^{\overline{\eta}} \left(\frac{\kappa (1-\sigma) H(\phi^*(\sigma, \eta))}{1 - (1-\kappa)(1-\sigma) H(\phi^*(\sigma, \eta))} \right) g(\eta) d\eta$$

4. Solve for the equilibrium probability of paying penalties through non-leniency enforcement σ^* which is a solution to the fixed point problem: $\sigma^* = \Psi(\sigma^*)$. If there are multiple fixed points, select the maximum. The equilibrium cartel rate is $C(\sigma^*)$.

Proof of Theorem 2. The proof has three steps. First, holding Y fixed, the threshold for stable collusion is shown to be lower with a leniency program: $\phi_{NL}(Y,\eta) > \phi_{\theta}(Y,\eta)$. When $\sigma > \theta$, which holds by supposition, the deviator has lower penalties by applying for leniency and this tightens the ICC and thus raises the threshold. Second, given $\phi_{NL}(Y,\eta) > \phi_{\theta}(Y,\eta)$ and the supposition that $\omega > \sigma$, it is shown that $\psi_{NL}(Y,\sigma,\eta) \geq \psi_{\theta}(Y,\sigma,\eta)$. That the collusive value function is weakly lower with a leniency program is due to two effects: i) $\phi_{NL}(Y,\eta) > \phi_{\theta}(Y,\eta)$ results in weakly shorter cartel duration with a leniency program; and ii) when there is a leniency program, expected penalties upon cartel collapse are $\omega\gamma(Y-\alpha\mu)$ rather than $\sigma\gamma(Y-\alpha\mu)$, and the former are higher when $\omega > \sigma$. Third, $\psi_{NL}(Y,\sigma,\eta) > \psi_{\theta}(Y,\sigma,\eta)$ implies a weakly lower fixed point with a leniency program - $Y^{*}_{NL}(\sigma,\eta) \geq Y^{*}_{\theta}(\sigma,\eta)$ - and, therefore, a weakly lower equilibrium threshold: $\phi^{*}_{NL}(\sigma,\eta) \geq \phi^{*}_{\theta}(\sigma,\eta)$. This proves the cartel rate is no higher with a leniency program. If, in addition, assumption A1 holds then $\phi^{*}_{NL}(\sigma,\eta) > \phi^{*}_{\theta}(\sigma,\eta)$ for a positive measure of values for η. From this result, one can then conclude that, holding σ fixed, the cartel rate is strictly lower with a leniency program.

Holding Y fixed, the threshold function for stable collusion is lower with a leniency program:

$$\begin{aligned}
& \phi_{NL}(Y,\sigma,\eta) - \phi_{\theta}(Y,\sigma,\eta) \qquad (1) \\
= \; & \frac{\delta(1-\sigma)(1-\kappa)(Y-\alpha\mu)}{(\eta-1)[1-\delta(1-\kappa)]} \\
& - \left(\frac{\delta(1-\sigma)(1-\kappa)(Y-\alpha\mu) - [1-\delta(1-\kappa)][\sigma - \min\{\sigma,\theta\}]\gamma(Y-\alpha\mu)}{(\eta-1)[1-\delta(1-\kappa)]} \right) \\
= \; & \frac{(\sigma-\theta)\gamma(Y-\alpha\mu)}{\eta-1} > 0
\end{aligned}$$

because $\sigma > \theta$. Using $\phi_{NL}(Y,\eta) > \phi_{\theta}(Y,\eta)$,

$$\begin{aligned}
& \psi_{NL}(Y,\sigma,\eta) - \psi_{\theta}(Y,\sigma,\eta) \\
= \; & \int_{\underline{\pi}}^{\phi_{NL}(Y,\sigma,\eta)} \{(1-\delta)\pi + \delta[(1-\sigma)Y + \sigma W] - (1-\delta)\sigma\gamma(Y-\alpha\mu)\} h(\pi)\, d\pi \\
& + \int_{\phi_{NL}(Y,\sigma,\eta)}^{\overline{\pi}} [(1-\delta)\alpha\pi + \delta W - (1-\delta)\sigma\gamma(Y-\alpha\mu)] h(\pi)\, d\pi \\
& - \int_{\underline{\pi}}^{\phi_{\theta}(Y,\sigma,\eta)} \{(1-\delta)\pi + \delta[(1-\sigma)Y + \sigma W] - (1-\delta)\sigma\gamma(Y-\alpha\mu)\} h(\pi)\, d\pi \\
& - \int_{\phi_{\theta}(Y,\sigma,\eta)}^{\overline{\pi}} [(1-\delta)\alpha\pi + \delta W - \omega\gamma(Y-\alpha\mu)] h(\pi)\, d\pi
\end{aligned}$$

and a few manipulations yields

$$\begin{aligned}
& \psi_{NL}(Y,\sigma,\eta)-\psi_{\theta}(Y,\sigma,\eta) \\
= & \int_{\phi_{\theta}(Y,\sigma,\eta)}^{\phi_{NL}(Y,\sigma,\eta)}\left[(1-\delta)(1-\alpha)\pi+\delta(1-\sigma)(Y-W)\right]h(\pi)\,d\pi \\
& +\int_{\phi_{\theta}(Y,\sigma,\eta)}^{\overline{\pi}}(1-\delta)(\omega-\sigma)\gamma(Y-\alpha\mu)h(\pi)\,d\pi.
\end{aligned} \tag{2}$$

Given $\omega>\sigma$, (2) is non-negative. If $\underline{\pi}\geq\phi_{NL}(Y,\sigma,\eta)\,(>\phi_{\theta}(Y,\sigma,\eta))$ or $(\phi_{NL}(Y,\sigma,\eta)>)\ \phi_{\theta}(Y,\sigma,\eta)\geq\overline{\pi}$ then the first of the two terms in (2) is zero; otherwise, it is positive. If $\phi_{\theta}(Y,\sigma,\eta)\geq\overline{\pi}$ then the second term is zero; otherwise, it is positive.

Since it has just been shown that $\psi_{NL}(Y,\sigma,\eta)\geq\psi_{\theta}(Y,\sigma,\eta)$ then $Y^{*}_{NL}(\sigma,\eta)\geq Y^{*}_{\theta}(\sigma,\eta)$. Given

$$\phi^{*}(\sigma,\eta)\equiv\max\left\{\min\left\{\phi(Y^{*}(\sigma,\eta),\sigma,\eta),\overline{\pi}\right\},\underline{\pi}\right\},$$

it follows that $\phi^{*}_{NL}(\sigma,\eta)\geq\phi^{*}_{\theta}(\sigma,\eta)$.

Next we want to show: if assumption A1 holds then $\phi^{*}_{NL}(\sigma,\eta)>\phi^{*}_{\theta}(\sigma,\eta)$ for a positive measure of values for η. If $\phi^{*}_{NL}(\sigma,\eta)<\overline{\pi}$ then either $\phi^{*}_{NL}(\sigma,\eta)>\underline{\pi}$ - so that $\phi^{*}_{NL}(\sigma,\eta)\in(\underline{\pi},\overline{\pi})$ - or $\phi^{*}_{NL}(\sigma,\eta)=\underline{\pi}$; and if $\phi^{*}_{NL}(\sigma,\eta)>\underline{\pi}$ then either $\phi^{*}_{NL}(\sigma,\eta)<\overline{\pi}$ - so that $\phi^{*}_{NL}(\sigma,\eta)\in(\underline{\pi},\overline{\pi})$ - or $\phi^{*}_{NL}(\sigma,\eta)=\overline{\pi}$. This results in two mutually exclusive cases: 1) there is a positive measure of values for η such that $\phi^{*}_{NL}(\sigma,\eta)\in(\underline{\pi},\overline{\pi})$; and 2) there is not a positive measure of values for η such that $\phi^{*}_{NL}(\sigma,\eta)\in(\underline{\pi},\overline{\pi})$ in which case there is a positive measure of values for η such that $\phi^{*}_{NL}(\sigma,\eta)=\underline{\pi}$ and a positive measure of values for η such that $\phi^{*}_{NL}(\sigma,\eta)=\overline{\pi}$.

In considering case (1), first note that

$$\phi^{*}_{NL}(\sigma,\eta)=\phi(Y^{*}_{NL}(\sigma,\eta),\sigma,\eta)>\phi_{\theta}(Y^{*}_{NL}(\sigma,\eta),\sigma,\eta)\geq\phi_{\theta}(Y^{*}_{\theta}(\sigma,\eta),\sigma,\eta), \tag{3}$$

where the equality follows from $\phi^{*}_{NL}(\sigma,\eta)\in(\underline{\pi},\overline{\pi})$, the strict inequality follows from $\phi_{NL}(Y,\sigma,\eta)>\phi_{\theta}(Y,\sigma,\eta)$, and the weak inequality follows from $Y^{*}_{NL}(\sigma,\eta)\geq Y^{*}_{\theta}(\sigma,\eta)$. (3) implies $\overline{\pi}>\phi_{\theta}(Y^{*}_{\theta}(\sigma,\eta),\sigma,\eta)$ and, therefore,

$$\phi^{*}_{\theta}(\sigma,\eta)=\max\left\{\phi_{\theta}(Y^{*}_{\theta}(\sigma,\eta),\sigma,\eta),\underline{\pi}\right\}. \tag{4}$$

(3)-(4) allow us to conclude: $\phi^{*}_{NL}(\sigma,\eta)>\phi^{*}_{\theta}(\sigma,\eta)$. Hence, for case (1), there is a positive measure of values for η for which $\phi^{*}_{NL}(\sigma,\eta)>\phi^{*}_{\theta}(\sigma,\eta)$. Under case (2), that $\phi^{*}_{NL}(\sigma,\eta)$ is weakly decreasing in (proof available on request) implies $\exists\widehat{\eta}_{NL}\in\left(\underline{\eta},\overline{\eta}\right)$ such that

$$\phi^{*}_{NL}(\sigma,\eta)=\begin{cases}\overline{\pi} & \text{if } \eta\in\left[\underline{\eta},\widehat{\eta}_{NL}\right]\\ \underline{\pi} & \text{if } \eta\in\left(\widehat{\eta}_{NL},\overline{\eta}\right]\end{cases}. \tag{5}$$

Note that, at the critical value $\widehat{\eta}_{NL}$,

$$\psi_{NL}(Y,\sigma,\widehat{\eta}_{NL})\leq Y,\ \forall Y\in[\alpha\mu,\mu], \tag{6}$$

for suppose not. Then $\exists Y'\in(\alpha\mu,\mu)$ such that $\psi_{NL}(Y',\sigma,\widehat{\eta}_{NL})>Y'$. By the continuity of ψ_{NL} in η, $\exists\xi>0$ such that $\psi_{NL}(Y',\sigma,\widehat{\eta}_{NL}+\xi)>Y'$ which implies

$Y^*_{NL}(\sigma, \widehat{\eta}_{NL} + \xi) > \alpha\mu$ and $\phi^*_{NL}(\sigma, \widehat{\eta}_{NL} + \xi) > \underline{\pi}$, but that contradicts (5). With (6) and $\psi_{NL}(Y, \sigma, \eta) > \psi_\theta(Y, \sigma, \eta)$, it follows $\exists \chi > 0$ such that $\psi_\theta(Y, \sigma, \widehat{\eta}_{NL}) < Y - \chi$ $\forall Y \in [\alpha\mu, \mu]$ which implies, by the continuity of ψ_θ in η, $\exists \widehat{\eta}_\theta < \widehat{\eta}_{NL}$ such that $\phi^*_\theta(\sigma, \eta) = \underline{\pi}$ iff $\eta > \widehat{\eta}_\theta$. We then have that there is a positive measure of values of η - specifically, $\eta \in [\widehat{\eta}_{NL}, \widehat{\eta}_\theta)$ - for which

$$\phi^*_{NL}(\sigma, \eta) = \overline{\pi} > \underline{\pi} = \phi^*_\theta(\sigma, \eta).$$

This concludes the proof that: if assumption A1 holds then $\phi^*_{NL}(\sigma, \eta) > \phi^*_\theta(\sigma, \eta)$ for positive measure of values for η.

Whether with or without a leniency program, if the threshold for a type-η industry is $\widetilde{\phi}(\sigma, \eta)$ then the cartel rate is

$$\int_{\underline{\eta}}^{\overline{\eta}} \left[\frac{\kappa(1-\sigma) H\left(\widetilde{\phi}(\sigma, \eta)\right)}{1 - (1-\kappa)(1-\sigma) H\left(\widetilde{\phi}(\sigma, \eta)\right)} \right] g(\eta)\, d\eta. \tag{7}$$

Note that the cartel rate is increasing in $\widetilde{\phi}(\sigma, \eta)$. Given it has been shown $\phi^*_{NL}(\sigma, \eta) \geq \phi^*_\theta(\sigma, \eta)$ $\forall \eta$, (7) implies $C_{NL}(\sigma) \geq C_\theta(\sigma)$. It has also been shown that: if there is a positive measure of values of η such that $\phi^*_{NL}(\sigma, \eta) < \overline{\pi}$ and a positive measure of values for η such that $\phi^*_{NL}(\sigma, \eta) > \underline{\pi}$ then there is a positive measure of values of η such that $\phi^*_{NL}(\sigma, \eta) > \phi^*_\theta(\sigma, \eta)$ and, therefore,

$$\frac{\kappa(1-\sigma) H\left(\phi^*_{NL}(\sigma, \eta)\right)}{1 - (1-\kappa)(1-\sigma) H\left(\phi^*_{NL}(\sigma, \eta)\right)} > \frac{\kappa(1-\sigma) H\left(\phi^*_\theta(\sigma, \eta)\right)}{1 - (1-\kappa)(1-\sigma) H\left(\phi^*_\theta(\sigma, \eta)\right)}. \tag{8}$$

As (8) holds for a positive measure of values of η, (7) implies $C_{NL}(\sigma) > C_\theta(\sigma)$. ■

The existence of a fixed point to $\Psi : [0,1] \to [0,1]$ is not immediate because there are two possible sources of discontinuity. Recall that $\phi^*(\sigma, \eta)$ depends on $Y^*(\sigma, \eta)$ which is the maximal fixed point to: $Y = \psi(Y, \sigma, \eta)$. Because of multiple fixed points to $\psi(Y, \sigma, \eta)$, $Y^*(\sigma, \eta)$ need not be continuous in σ and if $Y^*(\sigma, \eta)$ is discontinuous then $\phi^*(\sigma, \eta)$ is discontinuous which implies $H(\phi^*(\sigma, \eta))$ and $C(\sigma, \eta)$ from (12) in the paper are discontinuous. However, it is proven in Theorem 3 that these possible discontinuities in the integrand of Ψ do not create discontinuities in Ψ. The second possible source of discontinuity in Ψ is due to a discontinuity in expected penalties at $\sigma = \theta$. That discontinuity is present as long as $\theta \in (0,1)$ and, as a result, existence is established only when there is no leniency ($\theta = 1$) and full leniency ($\theta = 0$).

Proof of Theorem 3. When $\theta = 1$ then

$$\Psi(\sigma) = qrp\left(qr \int_{\underline{\eta}}^{\overline{\eta}} C(\sigma, \eta)\, g(\eta)\, d\eta \right), \tag{9}$$

and when $\theta = 0$ then

$$\begin{aligned} \Psi(\sigma) \quad = \quad & qrp\left(\lambda \int_{\underline{\eta}}^{\overline{\eta}} (1 - H(\phi^*(\sigma, \eta)))\, C(\sigma, \eta)\, g(\eta)\, d\eta \right. & (10) \\ & \left. + qr \int_{\underline{\eta}}^{\overline{\eta}} H(\phi^*(\sigma, \eta))\, C(\sigma, \eta)\, g(\eta)\, d\eta \right). \end{aligned}$$

To show that a fixed point exists for (9) and for (10), the proof strategy has two steps: 1) show that, for any value of σ, the integrand in these equations is continuous in σ except for a countable set of values of η; and 2) show that it follows from step 1 that Ψ is continuous. The proof will focus exclusively on proving that (10) has a fixed point as the method of proof is immediately applicable to the case of (9).[1]

Considering the integrand in (10), a discontinuity in

$$H\left(\phi^*\left(\sigma,\eta\right)\right)C\left(\sigma,\eta\right)g\left(\eta\right)=H\left(\phi^*\left(\sigma,\eta\right)\right)\left(\frac{\kappa\left(1-\sigma\right)H\left(\phi^*\left(\sigma,\eta\right)\right)}{1-\left(1-\kappa\right)\left(1-\sigma\right)H\left(\phi^*\left(\sigma,\eta\right)\right)}\right)g\left(\eta\right)$$

with respect to σ (or η) comes from $\phi^*\left(\sigma,\eta\right)$ being discontinuous, which comes from $Y^*\left(\sigma,\eta\right)$ being discontinuous. Let $\Delta\left(\sigma'\right)\subseteq\left[\underline{\eta},\overline{\eta}\right]$ be the set of η for which $Y^*\left(\sigma,\eta\right)$ is discontinuous at $\sigma=\sigma'$. We will show that $\Delta\left(\sigma\right)$ is countable.

Suppose $Y^*\left(\sigma,\eta\right)$ is discontinuous in σ at $\left(\sigma,\eta\right)=\left(\sigma',\eta'\right)$. Given $\psi\left(Y,\sigma,\eta\right)$ is continuous and $Y^*\left(\sigma,\eta\right)$ is the maximal fixed point to $\psi\left(Y,\sigma,\eta\right)$ then

$$\psi\left(Y,\sigma',\eta'\right)<Y,\forall Y\in\left(Y^*\left(\sigma',\eta'\right),\mu\right]. \tag{11}$$

If, in addition, $\exists\xi>0$ such that

$$\psi\left(Y,\sigma',\eta'\right)>Y,\forall Y\in\left[Y^*\left(\sigma',\eta'\right)-\xi,Y^*\left(\sigma',\eta'\right)\right)$$

then, by the continuity of $\psi\left(Y,\sigma,\eta\right)$ in σ, $Y^*\left(\sigma,\eta\right)$ is continuous at $\left(\sigma,\eta\right)=\left(\sigma',\eta'\right)$, contrary to our supposition. Hence, it must be the case that $\exists\xi>0$ such that

$$\psi\left(Y,\sigma',\eta'\right)\leq Y,\forall Y\in\left[Y^*\left(\sigma',\eta'\right)-\xi,Y^*\left(\sigma',\eta'\right)\right]. \tag{12}$$

Given that $\psi\left(Y,\sigma,\eta\right)$ is continuous and decreasing in η (proof available on request) then (11) and (12) imply

$$\psi\left(Y,\sigma',\eta'\right)<Y,\forall Y\in\left[Y^*\left(\sigma',\eta'\right)-\xi,\mu\right],\ \forall\eta>\eta'. \tag{13}$$

It follows from (13) that, $\forall\eta>\eta'$, all fixed points to ψ are bounded above by $Y^*\left(\sigma',\eta'\right)-\xi$:

$$Y^*\left(\sigma',\eta\right)<Y^*\left(\sigma',\eta'\right)-\xi,\ \forall\eta>\eta'.$$

Next define:

$$\varepsilon\left(\sigma',\eta'\right)\equiv Y^*\left(\sigma',\eta'\right)-\lim_{\eta\downarrow\eta'}Y^*\left(\sigma',\eta\right)$$

where $\varepsilon\left(\sigma',\eta'\right)$ measures the size of the discontinuity in $Y^*\left(\sigma',\eta\right)$ with respect to η at $\eta=\eta'$.

For each $\eta\in\Delta\left(\sigma'\right)$, there has then been associated an interval of length $\varepsilon\left(\sigma',\eta\right)$. Note that these intervals have a null intersection because $Y^*\left(\sigma,\eta\right)$ is non-increasing in η. Hence,

$$\sum_{\eta\in\Delta\left(\sigma'\right)}\varepsilon\left(\sigma',\eta\right)\leq\left(1-\alpha\right)\mu.$$

[1] When $\theta=1$, existence of a fixed point can also be established by showing that $\Psi\left(\sigma\right)$ is non-decreasing in σ and appealing to Tarski's Fixed Point Theorem. However, when $\theta<1$, it is generally not true that $\Psi\left(\sigma\right)$ is non-decreasing in σ $\forall\sigma$.

Given that a sum can only be finite if the number of elements which are positive is countable, it follows that $\Delta(\sigma')$ is countable. Hence, the set of values for η for which $Y^*(\sigma', \eta)$ is discontinuous in σ at $\sigma = \sigma'$ is countable. This completes the first step.

By Jeffrey (1925), given that $H(\phi^*(\sigma, \eta)) C(\sigma, \eta) g(\eta)$ and $(1 - H(\phi^*(\sigma, \eta))) C(\sigma, \eta) g(\eta)$ are bounded in (σ, η) on $[0, 1] \times [\underline{\eta}, \overline{\eta}]$ and are continuous at $\sigma = \sigma'$ for all $\eta \in [\underline{\eta}, \overline{\eta}]$ except for a countable set then

$$\int_{\underline{\eta}}^{\overline{\eta}} H(\phi^*(\sigma, \eta)) C(\sigma, \eta) g(\eta) \, d\eta$$

and

$$\int_{\underline{\eta}}^{\overline{\eta}} (1 - H(\phi^*(\sigma, \eta))) C(\sigma, \eta) g(\eta) \, d\eta$$

are continuous at $\sigma = \sigma'$. Given that p is a continuous function, it follows that

$$p\left(\lambda \int_{\underline{\eta}}^{\overline{\eta}} (1 - H(\phi^*(\sigma, \eta))) C(\sigma, \eta) g(\eta) \, d\eta + qr \int_{\underline{\eta}}^{\overline{\eta}} H(\phi^*(\sigma, \eta)) C(\sigma, \eta) g(\eta) \, d\eta\right)$$

is continuous in σ. Hence, Ψ in (10) is continuous in σ and maps $[0, 1]$ into itself; therefore, a fixed point exists. The same method of proof can be used to show that a fixed point to (9) exists. ■

Proof of Theorem 4. The first step is to show that, as the penalty multiple γ goes to zero, the cartel rate function is the same with and without a leniency program:

$$\lim_{\gamma \to 0} C_{NL}(\sigma) = \lim_{\gamma \to 0} C_L(\sigma), \ \forall \sigma.$$

The second step is to show that, as $\gamma \to 0$, non-leniency enforcement is weaker with a leniency program:

$$\lim_{\gamma \to 0} \sigma_{NL}^* > \lim_{\gamma \to 0} \sigma_L^*.$$

These two results together imply that the equilibrium cartel rate with a leniency program is higher than without a leniency program when $\gamma \simeq 0$.

For the first step, let us begin by considering the thresholds for stable collusion. Without a leniency program,

$$\phi_{NL}(Y, \sigma, \eta) = \frac{\delta(1 - \sigma)(1 - \kappa)(Y - \alpha\mu)}{(\eta - 1)[1 - \delta(1 - \kappa)]}$$

and, trivially,[2]

$$\lim_{\gamma \to 0} \phi_{NL}(Y, \sigma, \eta) = \frac{\delta(1 - \sigma)(1 - \kappa)(Y - \alpha\mu)}{(\eta - 1)[1 - \delta(1 - \kappa)]}.$$

[2]Recall that $\phi_{NL}(Y, \sigma, \eta)$ comes out of the ICC and is the market condition that makes a firm indifferent between colluding and cheating. That γ does not matter is because the expected penalty is the same whether a firm sets the collusive price or cheats and undercuts the collusive price set by the other firms.

With a full leniency program,

$$\phi_L(Y,\sigma,\eta) = \frac{\delta(1-\sigma)(1-\kappa)(Y-\alpha\mu) - [1-\delta(1-\kappa)]\sigma\gamma(Y-\alpha\mu)}{(\eta-1)[1-\delta(1-\kappa)]}$$

and

$$\lim_{\gamma\to 0}\phi_L(Y,\sigma,\eta) = \frac{\delta(1-\sigma)(1-\kappa)(Y-\alpha\mu)}{(\eta-1)[1-\delta(1-\kappa)]}.$$

Hence,

$$\lim_{\gamma\to 0}\phi_{NL}(Y,\sigma,\eta) = \lim_{\gamma\to 0}\phi_L(Y,\sigma,\eta). \tag{14}$$

Turning to the collusive value functions, we have without a leniency program:

$$\begin{aligned}\psi_{NL}(Y,\sigma,\eta) &= \int_{\underline{\pi}}^{\phi_{NL}(Y,\sigma,\eta)}[(1-\delta)\pi+\delta(1-\sigma)Y+\delta\sigma W]h(\pi)\,d\pi \\ &\quad +\int_{\phi_{NL}(Y,\sigma,\eta)}^{\overline{\pi}}[(1-\delta)\alpha\pi+\delta W]h(\pi)\,d\pi-(1-\delta)\sigma\gamma(Y-\alpha\mu),\end{aligned}$$

and with a full leniency program:

$$\begin{aligned}\psi_L(Y,\sigma,\eta) &= \int_{\underline{\pi}}^{\phi_L(Y,\sigma,\eta)}[(1-\delta)\pi+\delta(1-\sigma)Y+\delta\sigma W-(1-\delta)\sigma\gamma(Y-\alpha\mu)]h(\pi)\,d\pi \\ &\quad +\int_{\phi_L(Y,\sigma,\eta)}^{\overline{\pi}}[(1-\delta)\alpha\pi+\delta W-(1-\delta)\omega\gamma(Y-\alpha\mu)]h(\pi)\,d\pi.\end{aligned}$$

Using (14),

$$\begin{aligned}\lim_{\gamma\to 0}\psi_{NL}(Y,\sigma,\eta) &= \lim_{\gamma\to 0}\psi_L(Y,\sigma,\eta) \qquad (15)\\ &= \int_{\underline{\pi}}^{\phi_{NL}(Y,\sigma,\eta)}[(1-\delta)\pi+\delta(1-\sigma)Y+\delta\sigma W]h(\pi)\,d\pi \\ &\quad +\int_{\phi_{NL}(Y,\sigma,\eta)}^{\overline{\pi}}[(1-\delta)\alpha\pi+\delta W]h(\pi)\,d\pi.\end{aligned}$$

Generically, (15) implies

$$\lim_{\gamma\to 0}Y^*_{NL}(\sigma,\eta) = \lim_{\gamma\to 0}Y^*_L(\sigma,\eta). \tag{16}$$

(It is only generic because it requires that, in an ε-ball around $\gamma=0$, $Y^*_{NL}(\sigma,\eta)$ and $Y^*_L(\sigma,\eta)$ are continuous in γ.) It follows from (14) and (15) that:

$$\lim_{\gamma\to 0}\phi^*_{NL}(\sigma,\eta) = \lim_{\gamma\to 0}\phi^*_L(\sigma,\eta). \tag{17}$$

Given σ, the cartel rate without and with a leniency program, respectively, is:

$$C_{NL}(\sigma) = \int_{\underline{\eta}}^{\overline{\eta}} C_{NL}(\sigma,\eta)g(\eta)\,d\eta = \int_{\underline{\eta}}^{\overline{\eta}}\left[\frac{\kappa(1-\sigma)H(\phi^*_{NL}(\sigma,\eta))}{1-(1-\kappa)(1-\sigma)H(\phi^*_{NL}(\sigma,\eta))}\right]g(\eta)\,d\eta$$

$$C_L(\sigma) = \int_{\underline{\eta}}^{\overline{\eta}} C_L(\sigma,\eta)\, g(\eta)\, d\eta = \int_{\underline{\eta}}^{\overline{\eta}} \left[\frac{\kappa (1-\sigma) H(\phi_L^*(\sigma,\eta))}{1-(1-\kappa)(1-\sigma) H(\phi_L^*(\sigma,\eta))} \right] g(\eta)\, d\eta.$$

Using (17),

$$\lim_{\gamma\to 0} C_{NL}(\sigma) = \lim_{\gamma\to 0} C_L(\sigma). \tag{18}$$

To prove the second step, we want to first show that, when $\lambda > qr$ and $\gamma \simeq 0$,

$$\begin{aligned} & p\left(qr \int_{\underline{\eta}}^{\overline{\eta}} C_{NL}(\sigma,\eta)\, g(\eta)\, d\eta \right) \\ > \quad & p\left(\lambda \int_{\underline{\eta}}^{\overline{\eta}} (1-H(\phi_L^*(\sigma,\eta)))\, C_L(\sigma,\eta)\, g(\eta)\, d\eta + qr \int_{\underline{\eta}}^{\overline{\eta}} H(\phi_L^*(\sigma,\eta))\, C_L(\sigma,\eta)\, g(\eta)\, d\eta \right). \end{aligned} \tag{19}$$

Given p is strictly decreasing, (19) holds iff

$$\begin{aligned} & \lambda \int_{\underline{\eta}}^{\overline{\eta}} (1-H(\phi_L^*(\sigma,\eta)))\, C_L(\sigma,\eta)\, g(\eta)\, d\eta + qr \int_{\underline{\eta}}^{\overline{\eta}} H(\phi_L^*(\sigma,\eta))\, C_L(\sigma,\eta)\, g(\eta)\, d\eta \\ > \quad & qr \int_{\underline{\eta}}^{\overline{\eta}} C_{NL}(\sigma,\eta)\, g(\eta)\, d\eta \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \int_{\underline{\eta}}^{\overline{\eta}} (1-H(\phi_L^*(\sigma,\eta))) \left[\lambda C_L(\sigma,\eta) - qr C_{NL}(\sigma,\eta)\right] g(\eta)\, d\eta \\ > \quad & qr \int_{\underline{\eta}}^{\overline{\eta}} H(\phi_L^*(\sigma,\eta)) \left[C_{NL}(\sigma,\eta) - C_L(\sigma,\eta)\right] g(\eta)\, d\eta. \end{aligned} \tag{20}$$

Given (18), (20) holds as $\gamma \to 0$ iff

$$(\lambda - qr) \int_{\underline{\eta}}^{\overline{\eta}} (1-H(\phi_{NL}^*(\sigma,\eta)))\, C_{NL}(\sigma,\eta)\, g(\eta)\, d\eta > 0. \tag{21}$$

By the assumption in Theorem 4 (equation (13) in the paper),

$$\int_{\underline{\eta}}^{\overline{\eta}} (1-H(\phi_{NL}^*(\sigma,\eta)))\, C_{NL}(\sigma,\eta)\, g(\eta)\, d\eta > 0 \tag{22}$$

holds for $\sigma = \sigma_{NL}^*$. Given $\lambda > qr$ then (21) holds. We have shown that if $\lambda > qr$ then there exists $\widehat{\gamma} > 0$ such that if $\gamma \in [0, \widehat{\gamma}]$ then (19) holds, generically, in a small neighborhood of $\sigma = \sigma_{NL}^*$.

For when there is no leniency program, σ_{NL}^* is defined by:

$$\sigma_{NL}^* = qrp\left(qr \int_{\underline{\eta}}^{\overline{\eta}} C_{NL}(\sigma_{NL}^*,\eta)\, g(\eta)\, d\eta \right).$$

As it is the maximal fixed point then:

$$\sigma - qrp\left(qr\int_{\underline{\eta}}^{\overline{\eta}} C_{NL}\left(\sigma,\eta\right)g\left(\eta\right)d\eta\right) \gtreqqless 0 \text{ as } \sigma \gtreqqless \sigma^*_{NL}. \tag{23}$$

Hence, using (19), it follows from (23) that there exists $\widehat{\lambda} < 1$ and $\widehat{\gamma} > 0$ such that if $(\gamma,\lambda) \in [0,\widehat{\gamma}] \times \left[\widehat{\lambda},1\right]$ then $\exists\varepsilon > 0$ such that

$$\begin{aligned} & \sigma - qrp\left(\lambda\int_{\underline{\eta}}^{\overline{\eta}}\left(1-H\left(\phi^*_L\left(\sigma,\eta\right)\right)\right)C_L\left(\sigma,\eta\right)g\left(\eta\right)d\eta\right. \quad (24)\\ & \left.+qr\int_{\underline{\eta}}^{\overline{\eta}} H\left(\phi^*_L\left(\sigma,\eta\right)\right)C_L\left(\sigma,\eta\right)g\left(\eta\right)d\eta\right) \\ > \ & 0, \forall\sigma \geq \sigma^*_{NL} - \varepsilon. \end{aligned}$$

Given the continuity of

$$p\left(\lambda\int_{\underline{\eta}}^{\overline{\eta}}\left(1-H\left(\phi^*_L\left(\sigma,\eta\right)\right)\right)C_L\left(\sigma,\eta\right)g\left(\eta\right)d\eta + qr\int_{\underline{\eta}}^{\overline{\eta}} H\left(\phi^*_L\left(\sigma,\eta\right)\right)C_L\left(\sigma,\eta\right)g\left(\eta\right)d\eta\right)$$

in σ (see the proof of Theorem 3), (24) implies the maximal fixed point σ^*_L is less than $\sigma^*_{NL} - \varepsilon$. Given (18) and having just shown

$$\lim_{\gamma\to 0}\sigma^*_{NL} > \lim_{\gamma\to 0}\sigma^*_L,$$

it follows that

$$\lim_{\gamma\to 0} C_L\left(\sigma^*_L\right) > \lim_{\gamma\to 0} C_{NL}\left(\sigma^*_{NL}\right).$$

■

Proof of Theorem 6. Given $\sigma^* \in (0,\omega)$ and $\theta = 0$, by Theorem 2 we have that $C_{NL}\left(\sigma\right) \geq C_L\left(\sigma\right)$ and, when there is positive measure of values for η such that $\phi^*_{NL}\left(\sigma,\eta\right) < \overline{\pi}$ and a positive measure of values for η such that $\phi^*_{NL}\left(\sigma,\eta\right) > \underline{\pi}$, $C_{NL}\left(\sigma\right) > C_L\left(\sigma\right)$. To prove this theorem, it is then sufficient to show $\sigma^*_L > \sigma^*_{NL}$.

σ^*_{NL} and σ^*_L are defined by:

$$\sigma^*_{NL} = qrp\left(qr\int_{\underline{\eta}}^{\overline{\eta}} C_{NL}\left(\sigma^*_{NL},\eta\right)g\left(\eta\right)d\eta\right)$$

where

$$C_{NL}\left(\sigma,\eta\right) = \frac{\kappa\left(1-\sigma\right)H\left(\phi^*_{NL}\left(\sigma,\eta\right)\right)}{1-\left(1-\kappa\right)\left(1-\sigma\right)H\left(\phi^*_{NL}\left(\sigma,\eta\right)\right)},$$

and

$$\begin{aligned} \sigma^*_L \ &= \ p\left(\lambda\int_{\underline{\eta}}^{\overline{\eta}}\left(1-H\left(\phi^*_L\left(\sigma^*_L,\eta\right)\right)\right)C_L\left(\sigma^*_L,\eta\right)g\left(\eta\right)d\eta\right. \\ & \left.+qr\int_{\underline{\eta}}^{\overline{\eta}} H\left(\phi^*_L\left(\sigma^*_L,\eta\right)\right)C_L\left(\sigma^*_L,\eta\right)g\left(\eta\right)d\eta\right) \end{aligned}$$

where

$$C_L(\sigma,\eta) = \frac{\kappa(1-\sigma) H(\phi_L^*(\sigma,\eta))}{1-(1-\kappa)(1-\sigma) H(\phi_L^*(\sigma,\eta))}.$$

If $\phi_{NL}^*(\sigma,\eta) > (=) \phi_L^*(\sigma,\eta)$ then

$$H(\phi_{NL}^*(\sigma,\eta)) > (=) H(\phi_L^*(\sigma,\eta))$$

and

$$C_{NL}(\sigma,\eta) > (=) C_L(\sigma,\eta),$$

in which case,

$$H(\phi_{NL}^*(\sigma,\eta)) C_{NL}(\sigma,\eta) > (=) H(\phi_L^*(\sigma,\eta)) C_L(\sigma,\eta).$$

It is immediate that if

$$H(\phi_{NL}^*(\sigma,\eta)) C_{NL}(\sigma,\eta) \geq H(\phi_L^*(\sigma,\eta)) C_L(\sigma,\eta), \forall\eta \tag{25}$$

and

$$H(\phi_{NL}^*(\sigma,\eta)) C_{NL}(\sigma,\eta) > H(\phi_L^*(\sigma,\eta)) C_L(\sigma,\eta), \text{ for positive measure of } \eta \tag{26}$$

then

$$\int_{\underline{\eta}}^{\overline{\eta}} H(\phi_{NL}^*(\sigma,\eta)) C_{NL}(\sigma,\eta) g(\eta) d\eta > \int_{\underline{\eta}}^{\overline{\eta}} H(\phi_L^*(\sigma,\eta)) C_L(\sigma,\eta) g(\eta) d\eta. \tag{27}$$

(25) is always true and (26) is true when there is positive measure of values for η such that $\phi_{NL}^*(\sigma,\eta) < \overline{\pi}$ and a positive measure of values for η such that $\phi_{NL}^*(\sigma,\eta) > \underline{\pi}$.

Evaluate (27) at $\sigma = \sigma_{NL}^*$:

$$\int_{\underline{\eta}}^{\overline{\eta}} H(\phi_{NL}^*(\sigma_{NL}^*,\eta)) C_{NL}(\sigma_{NL}^*,\eta) g(\eta) d\eta > \int_{\underline{\eta}}^{\overline{\eta}} H(\phi_L^*(\sigma_{NL}^*,\eta)) C_L(\sigma_{NL}^*,\eta) g(\eta) d\eta. \tag{28}$$

Noting that σ_{NL}^* does not depend on λ, if λ is sufficiently small then it follows from (28):

$$\begin{aligned} & qr\int_{\underline{\eta}}^{\overline{\eta}} H(\phi_{NL}^*(\sigma_{NL}^*,\eta)) C_{NL}(\sigma_{NL}^*,\eta) g(\eta) d\eta \\ > \; & \lambda\int_{\underline{\eta}}^{\overline{\eta}} (1-H(\phi_L^*(\sigma_{NL}^*,\eta))) C_L(\sigma_{NL}^*,\eta) g(\eta) d\eta + \\ & qr\int_{\underline{\eta}}^{\overline{\eta}} H(\phi_L^*(\sigma_{NL}^*,\eta)) C_L(\sigma_{NL}^*,\eta) g(\eta) d\eta \end{aligned} \tag{29}$$

Given that p is decreasing then (29) implies (when λ is sufficiently small):

$$\begin{aligned}
& qrp\left(\lambda\int_{\underline{\eta}}^{\overline{\eta}}\left(1-H\left(\phi_L^*\left(\sigma_{NL}^*,\eta\right)\right)\right)C_L\left(\sigma_{NL}^*,\eta\right)g\left(\eta\right)d\eta\right. \\
& \left.+qr\int_{\underline{\eta}}^{\overline{\eta}}H\left(\phi_L^*\left(\sigma_{NL}^*,\eta\right)\right)C_L\left(\sigma_{NL}^*,\eta\right)g\left(\eta\right)d\eta\right) \\
> \; & qrp\left(qr\int_{\underline{\eta}}^{\overline{\eta}}H\left(\phi_{NL}^*\left(\sigma_{NL}^*,\eta\right)\right)C_{NL}\left(\sigma_{NL}^*,\eta\right)g\left(\eta\right)d\eta\right) \\
> \; & qrp\left(qr\int_{\underline{\eta}}^{\overline{\eta}}C_{NL}\left(\sigma_{NL}^*,\eta\right)g\left(\eta\right)d\eta\right)=\sigma_{NL}^*.
\end{aligned}$$

Hence,

$$\begin{aligned}
& qrp\left(\lambda\int_{\underline{\eta}}^{\overline{\eta}}\left(1-H\left(\phi_L^*\left(\sigma_{NL}^*,\eta\right)\right)\right)C_L\left(\sigma_{NL}^*,\eta\right)g\left(\eta\right)d\eta\right. \\
& \left.+qr\int_{\underline{\eta}}^{\overline{\eta}}H\left(\phi_L^*\left(\sigma_{NL}^*,\eta\right)\right)C_L\left(\sigma_{NL}^*,\eta\right)g\left(\eta\right)d\eta\right) \\
> \; & \sigma_{NL}^*
\end{aligned}$$

and thus $\sigma_L^* > \sigma_{NL}^*$.

In proving $\sigma_L^* > \sigma_{NL}^*$, the preceding analysis presumed $\omega > \sigma$. If, contrary to that presumption, $\sigma_L^* \geq \omega$ then the supposition that $\omega > \sigma_{NL}^*$ would again imply $\sigma_L^* > \sigma_{NL}^*$. ■

Proof of Theorem 7. Given that $\sigma = qrs$ and r,$s \in [0,1]$ (hence, are bounded), it is immediate that

$$\lim_{q\to 0}\sigma_{NL}^* = 0, \lim_{q\to 0}\sigma_L^* = 0,$$

which implies

$$\lim_{q\to 0}C_{NL}\left(\sigma_{NL}^*\right)=\lim_{\sigma\to 0}C_{NL}\left(\sigma\right),\;\; \lim_{q\to 0}C_L\left(\sigma_L^*\right)=\lim_{\sigma\to 0}C_L\left(\sigma\right).$$

To show the equilibrium cartel rate is lower with a leniency program, it is then sufficient to prove:

$$\lim_{\sigma\to 0}C_{NL}\left(\sigma\right) > \lim_{\sigma\to 0}C_L\left(\sigma\right). \tag{30}$$

Given $\theta = 0 < \omega$ then $\sigma \in (\theta, \omega)$ holds as $\sigma \to 0$ in which case Theorem 6 proves (30). ■

Proof of Theorem 8. If $C(\sigma) > 0$ then $\widehat{\eta}(\sigma) > \underline{\eta}$ and $Y^*(\sigma,\eta) > \alpha\mu$ $\forall\eta \in (1, \widehat{\eta}(\sigma)]$. Furthermore, since $Y^*(\sigma, \widehat{\eta}(\sigma)) > \alpha\mu$ and $Y^*(\sigma,\eta)$ is non-increasing in η (proof available on request) then

$$\lim_{\eta\to 1} Y^*(\sigma,\eta) > \alpha\mu.$$

Recall

$$\begin{aligned}
\phi(Y,\sigma,\eta) &= \frac{\delta(1-\sigma)(1-\kappa)(Y-\alpha\mu)-[1-\delta(1-\kappa)][\sigma-\min\{\sigma,\theta\}]\gamma(Y-\alpha\mu)}{(\eta-1)[1-\delta(1-\kappa)]} \\
&= \frac{\{\delta(1-\sigma)(1-\kappa)-[1-\delta(1-\kappa)][\sigma-\min\{\sigma,\theta\}]\gamma\}(Y-\alpha\mu)}{(\eta-1)[1-\delta(1-\kappa)]}
\end{aligned}$$

and

$$\phi^*(\sigma,\eta) \equiv \max\{\min\{\phi(Y^*(\sigma,\eta),\sigma,\eta),\overline{\pi}\},\underline{\pi}\}$$

where this encompasses both the case of a full leniency program ($\theta = 0$) and no leniency program ($\theta = 1$). Given that $Y^*(\sigma,\eta)$ is bounded above $\alpha\mu$ as $\eta \to 1$ then

$$\begin{aligned}
\lim_{\eta\to1}\phi(Y^*(\sigma,\eta),\sigma,\eta) &= \lim_{\eta\to1}\frac{\{\delta(1-\sigma)(1-\kappa)-[1-\delta(1-\kappa)][\sigma-\min\{\sigma,\theta\}]\gamma\}(Y^*(\sigma,\eta)-\alpha\mu)}{(\eta-1)[1-\delta(1-\kappa)]} \\
&= +\infty
\end{aligned}$$

and, therefore,

$$\lim_{\eta\to1}H(\phi^*(\sigma,\eta)) = \lim_{\eta\to1}H(\max\{\min\{\phi(Y^*(\sigma,\eta),\sigma,\eta),\overline{\pi}\},\underline{\pi}\}) = 1.$$

Thus, when η is close to one, if a stable cartel forms (that is, $\phi^*(\sigma,\eta) < \overline{\pi}$) then it is fully stable (that is, $\phi^*(\sigma,\eta) = \overline{\pi}$).

Next note that

$$C(\sigma,\eta) = \frac{\kappa(1-\sigma)H(\phi^*(\sigma,\eta))}{1-(1-\kappa)(1-\sigma)H(\phi^*(\sigma,\eta))}$$

and, therefore,

$$\lim_{\eta\to1}C(\sigma,\eta) = \lim_{\eta\to1}\frac{\kappa(1-\sigma)H(\phi^*(\sigma,\eta))}{1-(1-\kappa)(1-\sigma)H(\phi^*(\sigma,\eta))} = \frac{\kappa(1-\sigma)}{1-(1-\kappa)(1-\sigma)}.$$

We then have:

$$\begin{aligned}
&\lim_{\eta\to1}[C_L(\sigma_L^*,\eta)-C_{NL}(\sigma_{NL}^*,\eta)] \\
&= \frac{\kappa(1-\sigma_L^*)}{1-(1-\kappa)\left(1-\sigma_L^*\right)} - \frac{\kappa(1-\sigma_{NL}^*)}{1-(1-\kappa)\left(1-\sigma_{NL}^*\right)} \\
&= \kappa\left[\frac{(1-\sigma_L^*)[1-(1-\kappa)(1-\sigma_{NL}^*)]-(1-\sigma_{NL}^*)[1-(1-\kappa)(1-\sigma_L^*)]}{\left[1-(1-\kappa)\left(1-\sigma_L^*\right)\right]\left[1-(1-\kappa)\left(1-\sigma_{NL}^*\right)\right]}\right] \\
&= \frac{\kappa(\sigma_{NL}^*-\sigma_L^*)}{\left[1-(1-\kappa)\left(1-\sigma_L^*\right)\right]\left[1-(1-\kappa)\left(1-\sigma_{NL}^*\right)\right]}.
\end{aligned}$$

■

Proof of Theorem 9. Let us first show: if $\sigma \in (0,\omega)$ and $\widehat{\eta}_{NL}(\sigma) \in \left(\underline{\eta},\overline{\eta}\right)$ then $\widehat{\eta}_{L}(\sigma) < \widehat{\eta}_{NL}(\sigma)$. By the definition of $\widehat{\eta}$, if $\widehat{\eta} \in \left(\underline{\eta},\overline{\eta}\right)$ then there is a (maximal) fixed point in Y of $\psi(Y,\sigma,\eta)$ such that $Y > \alpha\mu$ for $\eta = \widehat{\eta}$ but not for $\eta > \widehat{\eta}$:

$$\begin{aligned} \exists Y^{*}(\sigma,\widehat{\eta}) &\in (\alpha\mu,\mu] \text{ such that } Y \leqq \psi(Y,\sigma,\widehat{\eta}) \text{ as } Y \geqq Y^{*}(\sigma,\widehat{\eta}) \\ \nexists Y &\in (\alpha\mu,\mu] \text{ such that } Y = \psi(Y,\sigma,\eta), \forall \eta \in (\widehat{\eta},\overline{\eta}]. \end{aligned} \tag{31}$$

Recall that

$$\psi(Y,\sigma,\eta) = \begin{cases} \int_{\underline{\pi}}^{\phi(Y,\sigma,\eta)} \{(1-\delta)\pi + \delta[(1-\sigma)Y + \sigma W] - (1-\delta)\sigma\gamma(Y-\alpha\mu)\} h(\pi)\,d\pi & \text{if } \sigma \leq \theta \\ + \int_{\phi(Y,\sigma,\eta)}^{\overline{\pi}} [(1-\delta)\alpha\pi + \delta W - (1-\delta)\sigma\gamma(Y-\alpha\mu)] h(\pi)\,d\pi & \\ & \\ \int_{\underline{\pi}}^{\phi(Y,\sigma,\eta)} \{(1-\delta)\pi + \delta[(1-\sigma)Y + \sigma W] - (1-\delta)\sigma\gamma(Y-\alpha\mu)\} h(\pi)\,d\pi & \text{if } \theta < \sigma \\ + \int_{\phi(Y,\sigma,\eta)}^{\overline{\pi}} [(1-\delta)\alpha\pi + \delta W - (1-\delta)\omega\gamma(Y-\alpha\mu)] h(\pi)\,d\pi & \end{cases}.$$

and

$$\phi(Y,\sigma,\eta) = \frac{\delta(1-\sigma)(1-\kappa)(Y-\alpha\mu) - [1-\delta(1-\kappa)][\sigma - \min\{\sigma,\theta\}]\gamma(Y-\alpha\mu)}{(\eta-1)[1-\delta(1-\kappa)]}.$$

Let us next argue that

$$\phi(Y^{*}(\sigma,\widehat{\eta}),\sigma,\widehat{\eta}) \in (\underline{\pi},\overline{\pi}]. \tag{32}$$

Obviously, $Y^{*}(\sigma,\widehat{\eta}) > \alpha\mu$ implies $\phi(Y^{*}(\sigma,\widehat{\eta}),\sigma,\eta) > \underline{\pi}$. If $\phi(Y^{*}(\sigma,\widehat{\eta}),\sigma,\eta) > \overline{\pi}$ then, by the continuity of $\phi(Y,\sigma,\eta)$ in η, it follows that $\exists\varepsilon > 0$ such that $\phi(Y^{*}(\sigma,\widehat{\eta}),\sigma,\eta) > \overline{\pi}$ $\forall\eta \in (\widehat{\eta},\widehat{\eta}+\varepsilon)$. Given that η affects $\psi(Y,\sigma,\eta)$ only through $\phi(Y,\sigma,\eta)$ - and recalling that $H(\overline{\pi}) = 1$ - then

$$\psi(Y^{*}(\sigma,\widehat{\eta}),\sigma,\widehat{\eta}) = \psi(Y^{*}(\sigma,\widehat{\eta}),\sigma,\eta)\,\forall\eta \in (\widehat{\eta},\widehat{\eta}+\varepsilon)$$

which implies $Y^{*}(\sigma,\widehat{\eta})$ is a fixed point to $\psi(Y,\sigma,\eta)$ $\forall\eta \in (\widehat{\eta},\widehat{\eta}+\varepsilon)$ which contradicts (31). We then conclude $\phi(Y^{*}(\sigma,\widehat{\eta}),\sigma,\eta) \leq \overline{\pi}$ and (32) is true.

Using (32) for when there is no leniency program, $\widehat{\eta}_{NL}(\sigma) \in \left(\underline{\eta},\overline{\eta}\right)$ implies $\phi_{NL}(Y^{*}_{NL}(\sigma,\widehat{\eta}_{NL}),\sigma,\widehat{\eta}_{NL}) \in (\underline{\pi},\overline{\pi}]$. Since ϕ is increasing in Y, it then follows:

$$\overline{\pi} > \phi_{NL}(Y,\sigma,\widehat{\eta}_{NL}), \forall Y \in [\alpha\mu, Y^{*}_{NL}(\sigma,\widehat{\eta}_{NL})). \tag{33}$$

In the proof of Theorem 2 it was shown: if $\sigma > 0$ then $\phi_{NL}(Y,\sigma,\eta) > \phi_{L}(Y,\sigma,\eta)$. Given it is assumed $\sigma > 0$, (33) then implies

$$\overline{\pi} \geq \phi_{NL}(Y,\sigma,\widehat{\eta}_{NL}) > \phi_{L}(Y,\sigma,\widehat{\eta}_{NL}), \forall Y \in [\alpha\mu, Y^{*}_{NL}(\sigma,\widehat{\eta}_{NL})] \tag{34}$$

Now consider:

$$
\begin{aligned}
& \psi_{NL}(Y,\sigma,\eta) - \psi_L(Y,\sigma,\eta) \\
= & \int_{\underline{\pi}}^{\phi_{NL}(Y,\sigma,\eta)} \{(1-\delta)\pi + \delta[(1-\sigma)Y + \sigma W] - (1-\delta)\sigma\gamma(Y-\alpha\mu)\} h(\pi)\, d\pi \\
& + \int_{\phi_{NL}(Y,\sigma,\eta)}^{\overline{\pi}} [(1-\delta)\alpha\pi + \delta W - (1-\delta)\sigma\gamma(Y-\alpha\mu)] h(\pi)\, d\pi \\
& - \int_{\underline{\pi}}^{\phi_L(Y,\sigma,\eta)} \{(1-\delta)\pi + \delta[(1-\sigma)Y + \sigma W] - (1-\delta)\sigma\gamma(Y-\alpha\mu)\} h(\pi)\, d\pi \\
& - \int_{\phi_L(Y,\sigma,\eta)}^{\overline{\pi}} [(1-\delta)\alpha\pi + \delta W - (1-\delta)\omega\gamma(Y-\alpha\mu)] h(\pi)\, d\pi
\end{aligned}
$$

After some simplifying steps:

$$
\begin{aligned}
& \psi_{NL}(Y,\sigma,\eta) - \psi_L(Y,\sigma,\eta) \\
= & \int_{\phi_L(Y,\sigma,\eta)}^{\phi_{NL}(Y,\sigma,\eta)} \{(1-\delta)(1-\alpha)\pi + \delta(1-\sigma)(Y-W) - (1-\delta)(\sigma-\omega)\gamma(Y-\alpha\mu)\} h(\pi)\, d\pi \\
& + \int_{\phi_{NL}(Y,\sigma,\eta)}^{\overline{\pi}} [(1-\delta)\alpha\pi + \delta W - (1-\delta)(\sigma-\omega)\gamma(Y-\alpha\mu)] h(\pi)\, d\pi.
\end{aligned}
$$

Given $\sigma \in (0,\omega)$ and using (34), we have:

$$
\psi_{NL}(Y,\sigma,\widehat{\eta}_{NL}) - \psi_L(Y,\sigma,\widehat{\eta}_{NL}) > 0. \tag{35}
$$

Next note that it follows from $\widehat{\eta}_{NL} \in (\underline{\eta},\overline{\eta})$ that:

$$
\begin{aligned}
\psi_{NL}(Y,\sigma,\widehat{\eta}_{NL}) &\leq Y \;\forall Y \in [\alpha\mu, Y^*_{NL}(\sigma,\widehat{\eta}_{NL})] \\
\psi_{NL}(Y,\sigma,\widehat{\eta}_{NL}) &< Y \;\forall Y \in (Y^*_{NL}(\sigma,\widehat{\eta}_{NL}),\mu].
\end{aligned}
$$

Using (35), this implies

$$
\begin{aligned}
\psi_L(Y,\sigma,\widehat{\eta}_{NL}) &< Y \;\forall Y \in [\alpha\mu, Y^*(\sigma,\widehat{\eta}_{NL})] \\
\psi_L(Y,\sigma,\widehat{\eta}_{NL}) &< Y \;\forall Y \in (Y^*(\sigma,\widehat{\eta}_{NL}),\mu],
\end{aligned}
$$

and, therefore, $\widehat{\eta}_L(\sigma) < \widehat{\eta}_{NL}(\sigma)$.

We have thus far shown: $\widehat{\eta}_L(\sigma^*_{NL}) < \widehat{\eta}_{NL}(\sigma^*_{NL})$. Given that $\widehat{\eta}(\sigma)$ is non-increasing in σ (proof available on request), if $\sigma^*_L \geq \sigma^*_{NL}$ then $\widehat{\eta}_L(\sigma^*_L) \leq \widehat{\eta}_L(\sigma^*_{NL})$ which then implies $\widehat{\eta}_L(\sigma^*_L) < \widehat{\eta}_{NL}(\sigma^*_{NL})$. ∎

2 Appendix B: Numerical Methods[3]

There are 9 parameters in the general model: n, α, ω, θ, κ, δ, q, γ, and λ. The baseline simulation assumes: $(n, \alpha, \omega, \theta, \kappa, \delta, q, \gamma, \lambda) = (4, 0, .75, 0 \text{ or } 1, .05, .85, .2, .5, 1)$, where $\theta = 0$ with leniency program and $\theta = 1$ without leniency program.

For the probability of conviction function, we consider two functional forms:

$$p(\lambda L + R) = \begin{cases} \max\{c - m(\lambda L + R), 0.05\}, & \text{where } c < 1, \\ \frac{\tau}{\xi + \upsilon(\lambda L + R)^\rho}, & \text{where } \upsilon > 0,\ \rho \geq 1,\ \tau \in (0,1],\ \xi \geq \tau \end{cases}$$

For the first specification, the probability decreases linearly with the caseload until it reaches its minimum value of 0.05. The second specification assumes a concave then convex relationship between caseload and the probability of success. The baseline simulation assumes $(c, m) = (.8, 40)$ for the linear specification and $(\tau, \xi, \upsilon, \rho) = (1, 1, 1000, 1.4)$ for the non-linear specification.

We assume a log-normal distribution, $LN(\mu, \sigma^2)$, for the two distributions, $H(\pi)$ and $G(\eta)$, where $(\mu, \sigma) = (0, 1.5)$ for $H(\pi)$ and $(\mu, \sigma) = (1, 1.5)$ for $G(\eta)$. The lower and upper bounds for the distributions are: $(\underline{\pi}, \overline{\pi}) = (1, \infty)$,= and $(\underline{\eta}, \overline{\eta}) = (1.1, \infty)$.

The numerical problem has a nested structure. Given a value of r, the underlying problem is to find a fixed point, $\sigma^*(r)$, to $\sigma = q \times r \times p(\lambda L(\sigma) + R(\sigma))$, where $L(\sigma)$ is the mass of cartel cases generated by the leniency program and $R(\sigma)$ is the mass of non-leniency cartel cases.

The procedure for finding $\sigma^*(r)$ begins by specifying an initial value for σ. For each η, we need to solve for a fixed point determining the collusive value: $Y^*(\sigma, \eta) = \psi(Y^*(\sigma, \eta), \sigma, \eta)$. As there may be multiple fixed points, the Pareto criterion is used which selects the largest fixed point. Since $\psi(Y, \sigma, \eta)$ is increasing and $\psi(\mu, \sigma, \eta) < \mu$ then, by setting $Y^0 = \mu$ and iterating on $Y^{t+1} = \psi(Y^t, \sigma, \eta)$, this process converges to the largest fixed point.

In computing the stationary distribution of cartels, we need to take the step of computationally searching for $\widehat{\eta}(\sigma)$ which is the smallest industry type for which collusion is not incentive compatible for any market condition. $\widehat{\eta}(\sigma)$ is defined by: $Y^*(\sigma, \eta) > \alpha\mu$ for $\underline{\eta} < \eta \leq \widehat{\eta}(\sigma)$ and $Y^*(\sigma, \eta) = \alpha\mu$ for $\eta > \widehat{\eta}(\sigma)$. To perform this step, we set $\underline{\eta} = 1.1$ and $\overline{\eta} = 10$ and use a 1,000 element finite grid of values for η, denoted $\Gamma(\underline{\eta}, \overline{\eta})$. $\widehat{\eta}(\sigma)$ is located by applying the iterative bisection method on $\Gamma(\underline{\eta}, \overline{\eta})$. As part of the bisection method, $\overline{\eta}$ needs to be set at a sufficiently high value so that $Y^*(\sigma, \overline{\eta}) = \alpha\mu$. Once having identified $\widehat{\eta}(\sigma)$ and using $Y^*(\sigma, \eta)$, $\phi^*(\sigma, \eta)$ is calculated for a finite grid over $[\underline{\eta}, \widehat{\eta}(\sigma)]$. These values are then used in computing $L(\sigma)$ and $R(\sigma)$. The integration uses the Newton-Cotes quadrature method with the trapezoid rule (see Miranda and Fackler, 2002).

Choosing an initial value for σ and using our derived expressions for $L(\sigma)$ and $R(\sigma)$, we then compute: $\widehat{\sigma}(r) = q \times r \times p(\lambda L(\sigma) + R(\sigma))$. After specifying a tolerance level ϵ, if $|\sigma - \widehat{\sigma}(r)| > \epsilon$ then a new value for σ is selected using the iterative bisection

[3] The Mathematica code that generates the equilibrium cartel rates for the baseline case is available at: http://academic.csuohio.edu/changm/main/research/papers/CLPcodeA.pdf.

method. Note that once a new value for σ is specified, the entire preceding procedure must be repeated. This procedure is repeated until the process converges to the fixed point value of $\sigma^*(r)$ such that $|\sigma^*(r) - \widehat{\sigma}(r)| \leq \epsilon$. ϵ is set at .0002.

Given the equilibrium probability of paying penalties, $\sigma^*(r)$, we can calculate the equilibrium cartel rate, $C(\sigma^*(r))$, mass of leniency cases, $L(\sigma^*(r))$, and mass of non-leniency cases, $R(\sigma^*(r))$.

In order to reduce the number of parameters, we solve for r by setting it to minimize the equilibrium rate of cartels, $C(\sigma^*(r))$. Denoting this value as r^*, it is numerically derived by allowing $r \in \{0, .1, ..., 1\}$ and performing the procedures described above for each of these values to identify the one that generates the minimum cartel rate.

In addition to the baseline parameter values, we considered a wide variety of parameter values off of the baseline in order to check for the robustness of the main properties identified in the paper. Specifically, for both the linear and non-linear $p(\lambda L + R)$, we considered $\gamma \in \{0.7, 0.8, 0.9\}$. Further robustness checks were performed for the non-linear $p(\lambda L + R)$ for the following parameter values off of the baseline: $\rho \in \{1.2, 1.4, 1.6\}$, $\gamma \in \{0.3, 0.7, 2.0\}$, $\lambda \in \{0.6, 0.8\}$, $\upsilon \in \{100, 500\}$, $n = 2$ (and, hence, $\omega = 0.5$ or 1),[4] $\alpha \in \{0.2, 0.5\}$, $\kappa = 0.1$, and $\delta \in \{0.75, 0.95\}$. For all these parameter values, the numerical results are consistent with the properties stated in the paper: i) a leniency program can lower or raise the cartel rate; ii) the change in average cartel duration from a leniency program is decreasing in the industry type, η; and iii) $\widehat{\eta}$ is (generally) lower when there is a leniency program: $\widehat{\eta}_L < \widehat{\eta}_{NL}$.

[4] Note that $\omega = \frac{n-1+\theta}{n}$, where $\theta = 0$ with a leniency program and $\theta = 1$ without a leniency program. Hence, for $n = 2$, $\omega = 0.5$ with a leniency program and $\omega = 1$ without a leniency program.