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# Co-evolution of firms and consumers and the implications for market dominance

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**Abstract**

Consider a setting in which firms randomly discover new ideas that affect their products or services and implement favorable ones. At the same time that firms are adapting their offerings, consumers are searching among firms for the best match. It is shown that implicit in these dual dynamics is an increasing returns mechanism which can result in one firm dominating the market in the long run. The conditions under which there is sustained market dominance are characterized.


*JEL classification:* L1

*Keywords:* Market dominance; Innovation; Search

## 1. Introduction

Consider the following scenarios:

- Stores in a geographic market compete through their practices. Upon discovery of a new practice, a store manager evaluates the profitability of its adoption and decides whether or not to implement it. At the same time that stores' practices are evolving, consumers are searching among stores to find the one whose practices best conform to their preferences.
- Firms compete by modifying their products. Brand managers discover new product attributes which they adopt and sell in test markets. Those modifications that seem to

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work are retained and rolled out for the general market. At the same time, consumers are trying different products to find the best match.
- Internet sites compete by upgrading their site. Through online surveys and the tracking of clickstream behavior of those who visit their site, an online company learns about the preferences of visitors. Based on the information they have collected, a site evaluates new ideas and implements those that seem to meet the needs of their visitors. Simultaneously, consumers are surfing among sites to find the one they like best.

What are the implications of these dual dynamics – firms adapting their offerings and consumers sorting themselves among firms – for market dominance? If one firm initially has a better store or product or Internet site and thereby attracts a bigger share of the market, does it have a higher likelihood of being dominant in the future? If market dominance is achieved, how easily is it sustained? How does the rate of consumer experimentation affect the persistence of market dominance?

In addressing these questions, this paper makes two contributions. First, it identifies a new source of increasing returns predicated on the property that a firm's current customer base influences what innovations it adopts. The right customer mix leads a firm to adopt the right kind of ideas which induces consumer sorting that generates an even better customer mix leading the firm to adopt even better ideas. While this feedback system is based upon a firm's customer mix, as opposed to market share as in most other increasing returns mechanisms, this will ultimately lead to dominance as measured by market share. The second contribution is exploring when this increasing returns mechanism generates sustained market dominance – one firm persistently having a higher market share. Analysis is performed on two models. In the first model, firms' offerings are differentiated horizontally and innovation takes the form of a new set of attributes in this space. We show that, regardless of the rate of consumer experimentation, sustained market dominance can occur and is a more robust phenomenon than a symmetric market outcome. The model is then adapted to also allow the quality of firms' offerings to differ and be stochastic. If the maximum quality differential is sufficiently low, the possibility of sustained market dominance persists. If it is sufficiently high then sustained market dominance does not occur so that the identity of the market leader never gets locked in.

The increasing returns mechanism described in this paper is quite distinct from previously identified mechanisms. Learning-by-doing creates increasing returns because higher cumulative production results in lower marginal cost which induces the firm to price lower and produce more and this higher output further increases its cost advantage (see, for example, Cabral and Riordan, 1994). Another well-known source of increasing returns is associated with network externalities which a product possesses when its value to a consumer is increasing in how many other consumers use it. A firm that initially has a high share of users then has a more appealing product. This causes new consumers to adopt it at a higher rate which results in an even higher share of users in the future (see, for example, Katz and Shapiro, 1985; Farrell and Saloner, 1986). A third source of increasing returns is identified by Bagwell et al. (1997). Motivated by retail chains, they consider a setting in which a firm with higher sales has a greater incentive to invest in reducing marginal cost which leads it to set a lower price, thereby generating yet higher sales and a yet greater incentive to engage in cost-reducing investment.

## 2. Model

There are two firms: firms 1 and 2. At any point in time, a firm has a location that represents the attributes of its product or service. Let $x_i^{t-1}$ denote the attributes of firm $i$ at the start of period $t$. $x_i^{t-1}$ is restricted to lie in $X$ which is a finite subset of [0,1]. However, for purposes of the ensuing analysis, many of the functions in this section that depend on a firm's attributes will be defined $\forall x_i^{t-1} \in [0,1]$.[1] Time is discrete and unbounded so that $t=1,2,\ldots$ There is a continuum of consumers who have preferences over attributes with each consumer being defined by an ideal set of attributes. For simplicity, there are only two types of consumers. A type 0 consumer's ideal location is 0 and a type 1 consumer's ideal location is 1. A fraction $\alpha \in (0.5,1)$ of consumers are type 0. The type 0 consumer should be thought of as the typical consumer in this market and type 1 consumers as representing more of a niche sub-market.

At any point in time, a consumer is *loyal* to one of the firms which means buying from it with probability $1-\rho$ and buying from the other firm with probability $\rho \in (0,\frac{1}{2})$. One can think of $\rho$ as the rate of consumer search but also as being driven by exogenous forces disturbing a consumer's routine; for example, a consumer might happen to be near his less favored store or clicks a link to an Internet site while surfing.[2]

The profit to a firm with attributes $x$ generated by a type $k$ customer is specified to be $g(|k-x|)$ which is assumed to be a decreasing strictly concave function of $|k-x|$:

(A1) $g:[0,1] \to \Re_+$ is twice continuously differentiable.
(A2) $g'(0)=0, g'(d)<0\ \forall d \in (0,1]$, and $g''(d)<0\ \forall d \in [0,1]$.

Let $\pi(x,w(0),w(1)) : [0,1]\times[0,\alpha]\times[0,1-\alpha] \to \Re_+$ denote the profit to a firm when its location in attribute space is $x$ and it has a mass $w(0)$ of loyal type 0 customers and a mass $w(1)$ of loyal type 1 customers:[3]

$$\pi(x,w(0),w(1)) = [(1-\rho)w(0)+\rho(\alpha-w(0))]\,g(x) + [(1-\rho)w(1)+\rho(1-\alpha-w(1))]\,g(1-x). \tag{1}$$

[1] Specifying $X$ to be finite will allow us to use results from the theory of finite Markov chains. Results have also been derived when locations lie in [0,1] and are qualitatively similar though with more complicated proofs; see Harrington and Chang (2001).

[2] Results are robust to allowing $\rho$ to vary over time, either deterministically or stochastically. What is important is that $\rho$ is bounded above zero and below $\frac{1}{2}$.

[3] The astute reader will notice that a firm's current profit depends only on its current attributes and not on its rival's. While a rival's past attributes will influence whether a consumer comes to a firm – thus determining $(w(0),w(1))$ – once those consumers are there, it is the firm's attribute that determines how much profit the firm earns from those consumers.

A firm with $w(0)$ loyal type 0 customers finds a fraction $1-\rho$ of them buying from it and a fraction $\rho$ of the $\alpha - w(0)$ type 0 consumers who are loyal to the other firm. The total mass of type 0 consumers buying is then $[(1-\rho)w(0)+\rho(\alpha - w(0))]$ and from each of them the firm earns profit of $g(x)$. Similarly, one can explain the profit generated by type 1 consumers.

Firm $i$ enters period $t$ with attributes $x_i^{t-1}$. The discovery of alternative attributes is presumed to be an act of creativity. Contrary to the usual assumption that the attribute space is known, we assume that it is unknown and innovation involves identifying points in that space. More specifically, in each period, a firm comes up with a new set of attributes with probability $\omega \in (0,1)$.[4] The idea for period $t$ is denoted $y_i^t$ and is drawn according to a probability distribution with full support on $X$.

Given a new idea, firm $i$ decides either to discard it, in which case $x_i^t = x_i^{t-1}$ so that it maintains its current attributes, or to adopt it, in which case $x_i^t = y_i^t$.[5] A crucial feature to our model is how we specify the manner in which this decision is made. The firm is faced with a complex dynamic problem in that it will be receiving many ideas over time from an unknown space while operating in a perpetually changing environment as its competitor alters the character of its product and consumers switch loyalties. While a firm's behavior may be well approximated by an equilibrium strategy, such is not obvious. Rather than pursue that route, we have chosen an alternative approach by assuming that firms deploy heuristics – decision rules that, in a simple manner, condition on only part of an agent's information set. It is well documented that agents deploy heuristics when faced with complex environments.[6]

In guiding the specification of a firm's heuristic, we draw on our reading of the literature which suggests that retailers in consumer markets think about strategy in terms of satisfying some targeted group of consumers.[7] In each period, a firm is assumed to have a target customer base. A new idea is adopted if it satisfies that base in the sense of generating more profit from it. Otherwise, the idea is discarded. Rather than specify a specific rule of that form, we consider a wide class of such rules that is defined by the target customer base depending on the actual loyal customer base but not on firms' current attributes and, in this manner, uses limited information about the environment. A firm with loyal customer base $(w(0), w(1))$ is defined to have a target customer base comprised of the type 0 consumers with mass $\theta_0(w(0), w(1))$ and type 1 consumers with mass $\theta_1(w(0), w(1))$. Two restrictions are placed on this class of heuristics. By A4, the target customer base is always strictly positive so that no consumer type is ignored. This seems compelling since $\rho > 0$ implies that a firm will always have both consumer types buying from it. A5 requires that more loyal consumers of, say, type 0 raises the target base for type 0 consumers and (weakly)

[4] Assuming $\omega < 1$ simplifies some steps in the proofs. All results go through if $\omega = 1$.

[5] Though firms are not permitted to recall previously discovered ideas, if memory is bounded, so a firm could only retain some maximal number of ideas, our results should still be true.

[6] There is a very large literature here that would take us too far afield to seriously cover. A useful point of departure for interested readers is Gigerenzer et al. (1999).

[7] "The retailer should have a fully developed marketing strategy, which should include the specific target market. A target market is the group or groups of customers that the retailer is seeking to serve." (Dunne and Lusch, 1999, p. 50). Also, see Kotler (1997, Chapter 2).

lowers the target base for type 1 consumers. Thus, a firm's target set of consumers is responsive to its current customer base in an intuitively reasonable manner:

(A4) $\theta_i : [0,\alpha] \times [0,1-\alpha] \to \Re_{++}$ is continuously differentiable, $i \in \{0,1\}$.
(A5) $\partial\theta_i/\partial w(i) > 0$ and $\partial\theta_i/\partial w(j) \leqslant 0$ $(j \neq i)$, $i,j \in \{0,1\}$.

A firm's adoption decision regarding a new idea is based on *virtual profit* which is defined to be the profit based on the target customer base:

$$\tilde{\pi}(x, w(0), w(1)) = \theta_0(w(0), w(1))g(x) + \theta_1(w(0), w(1))g(1-x),$$

where $\tilde{\pi} : [0,1] \times [0,\alpha] \times [0,1-\alpha] \to \Re_+$. In Lemma 1, $\phi(w(0), w(1))$ is defined to be the location that maximizes virtual profit when the choice set for $x$ is [0,1]. It shows that the optimal firm location is well defined and is decreasing in the mass of type 0 loyal customers and increasing in the mass of type 1 loyal customers. Proofs are in Appendix B.

**Lemma 1.** $\exists\phi : [0,\alpha] \times [0,1-\alpha] \to [0,1]$ *such that*

$$\phi(w(0), w(1)) \in \arg\max \tilde{\pi}(x, w(0), w(1)). \tag{2}$$

$\phi$ *is unique*, $\partial\phi/\partial w(0) < 0$, *and* $\partial\phi/\partial w(1) > 0$.

Define $\underline{\phi} \equiv \phi(\alpha, 0)$ and $\bar{\phi} \equiv \phi(0, 1-\alpha)$ as the optimal location from [0,1] when a firm's loyal customers are all of the type 0 consumers and all of the type 1 consumers, respectively. By Lemma 1, it follows that $\phi(w(0), w(1)) \in [\underline{\phi}, \bar{\phi}]\,\forall\,(w(0), w(1))$. To save on notation, let $\beta^t$ and $\gamma^t$ denote the mass of type 0 consumers and type 1 consumers, respectively, that are loyal to firm 1 in period $t$. Firm 1's virtual profit in period $t$ can then be represented as $\tilde{\pi}_1(x_1^t, \beta^t, \gamma^t) \equiv \tilde{\pi}(x_1^t, \beta^t, \gamma^t)$ and firm 2's by $\tilde{\pi}_2(x_2^t, \beta^t, \gamma^t) \equiv \tilde{\pi}(x_2^t, \alpha - \beta^t, 1 - \alpha - \gamma^t)$. Finally, define $\phi_1(\beta^t, \gamma^t) \equiv \phi(\beta^t, \gamma^t)$ and $\phi_2(\beta^t, \gamma^t) \equiv \phi(\alpha - \beta^t, 1 - \alpha - \gamma^t)$.

Recall that a firm's attributes is restricted to being in $X$ where $X$ is a finite subset of [0,1] . For convenience, it is assumed that $\underline{\phi}, \bar{\phi}, \hat{\phi}, 0, 1 \in X$ where $\hat{\phi} \equiv \phi(\alpha/2, (1-\alpha)/2)$ is the optimal location when firms equally share the market. Generally, we want to think of $X$ as being fairly dense with the finiteness introduced to avoid the complications associated with an uncountable state space.

Given a target customer base, a new idea, $y_i^t$, is adopted if and only if it raises virtual profit. The dynamic on firm practices is then

$$x_i^t = \begin{cases} x_i^{t-1} & \text{if } \tilde{\pi}_i(x_i^{t-1}, \beta^t, \gamma^t) \geqslant \tilde{\pi}_i(y_i^t, \beta^t, \gamma^t), \\ y_i^t & \text{if } \tilde{\pi}_i(x_i^{t-1}, \beta^t, \gamma^t) < \tilde{\pi}_i(y_i^t, \beta^t, \gamma^t). \end{cases} \tag{3}$$

This class of heuristics encompasses many natural rules. One such rule is myopic hillclimbing – adopt an idea if it raises current profit – which is the case when the target base is the current customer base:

$$\theta_0(w(0), w(1)) = [(1-\rho)w(0) + \rho(\alpha - w(0))],$$

$$\theta_1(w(0), w(1)) = [(1-\rho)w(1) + \rho(1 - \alpha - w(1))]. \tag{4}$$

Although myopic, this heuristic is a plausible response to the dynamic problem at hand; founded on the idea that a firm should work to retain those consumers who are already loyal rather than pursue the riskier strategy of alienating them in order to attract other consumer types. Indeed, there is considerable emphasis in the business strategy literature on customer retention.[8] Furthermore, case studies document firms focusing their resources on developing innovations that serve existing customers. For several industries including the hard disk drive and mechanical excavator industries, Christensen (1997) argues that the reticence of leading firms to adopt drastic new technologies was due to them having "well-developed systems for killing ideas that their customers do not want" (p. xix). It was further argued that successful firms had learned to listen and respond to the needs of their existing customers and avoid projects that would not serve them but rather some other customer base.[9] It is also easy to see how this heuristic could be executed. A manager could implement a new idea for a period of experimentation which serves to reveal its profitability. What is implicitly assumed is that the length of the period of experimentation is very small so that firms engage in *virtual experimentation* (Gale and Rosenthal, 1999). One might further argue that those consumers who buy or visit a firm may be the best source of information about what are worthwhile ideas which would make (4) quite plausible.

Alternatively, a firm may seek to modify its practices so that they serve the broader market population. This can be encompassed by having the target customer base be a weighted average of their loyal (or current) customer base and the market population. The appeal of this heuristic is that it balances the short-run need to earn profit from the existing customer base with the long-term goal of attracting a broader market segment. Implementation might be achieved by using exit surveys of existing consumers along with the creation of focus groups based on demographic information on the local population.

Given this heuristic for judging ideas, it is straightforward to characterize the set of acceptable ideas. Let us initially do this when the set of ideas is [0,1]. With current location $x$ and loyal customers $(w(0), w(1))$, it follows from the strict concavity of $\tilde{\pi}$ that there is a connected set of locations that yields at least as high a level of virtual profit as is achieved with $x$. One extreme point of this set is $x$. The other extreme point, denoted $\psi(x, w(0), w(1))$, is defined by $\tilde{\pi}(\psi(x, w(0), w(1)), w(0), w(1)) = \tilde{\pi}(x, w(0), w(1))$ (see Fig. 1).[10] Define $\psi_1(x, \beta, \gamma) \equiv \psi(x, \beta, \gamma)$ and $\psi_2(x, \beta, \gamma) \equiv \psi(x, \alpha - \beta, 1 - \alpha - \gamma)$. The set of acceptable ideas in period $t$ is

$$[\min\{\psi_i(x_i^{t-1}, \beta^t, \gamma^t), x_i^{t-1}\}, \max\{\psi_i(x_i^{t-1}, \beta^t, \gamma^t), x_i^{t-1}\}].$$

[8] Reichheld and Sasser (1990) argue that the customer defection rate has a major impact on profit and that a firm should strive to have zero defections. An Arthur Andersen study found that it cost 5 – 15 times as much as to attract new consumers and that a 5% increase in customer retention can increase profits by 25 – 40% (*Chain Store Age*, November 1995, p. 88). Whether or not one finds these estimates meaningful (as sample selection bias is probably a serious problem), it does say something about how firms *perceive* their environment and thus what types of heuristics they may deploy.

[9] Another example is Lowe's decision to go from 20 000 square foot stores to 100 000 square foot stores which was purportedly based on an exit survey of 2400 customers (*Forbes*, December 18, 1995, pp. 116 – 117).

[10] When $\nexists x' \in [0,1]$ such that $\tilde{\pi}(x', w(0), w(1)) = \tilde{\pi}(x, w(0), w(1))$ then $\psi = 0$ if $\phi < x$ and $\psi = 1$ if $\phi > x$.

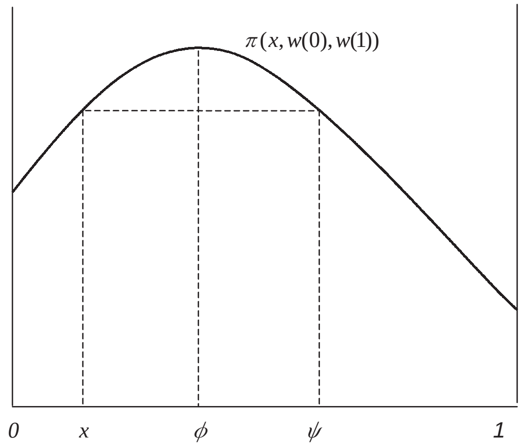


Fig. 1.

Of course, a firm is restricted to ideas in $X$ so that the set of acceptable and feasible ideas is

$$X \cap [\min\{\psi_i(x_i^{t-1}, \beta^t, \gamma^t), x_i^{t-1}\}, \max\{\psi_i(x_i^{t-1}, \beta^t, \gamma^t), x_i^{t-1}\}].$$

Next consider the equation of motion on a firm's loyal customers. A consumer who is loyal to firm $i$ in period $t$ and buys from firm $i$ in period $t$ is assumed to remain loyal. A type $k$ consumer who is loyal to firm $i$ in period $t$ and buys from store $j$ in period $t$ remains loyal to firm $i$ if $|x_i^{t-1} - k| < |x_j^t - k|$ and switches to being loyal to firm $j$ if $|x_i^{t-1} - k| > |x_j^t - k|$. When $|x_i^{t-1} - k| = |x_j^t - k|$ then 50% of such consumers switch loyalty. The idea is that a consumer's loyalty is based on how close a firm's product is to the consumer's ideal product. If a consumer loyal to firm $i$ bought from it in period $t-1$ but experimented with the other firm in period $t$ then the consumer is assumed to make this judgement by comparing their most recent experiences. To ensure that this is the preceding period for a consumer's favored firm, it is assumed that if a consumer experimented in period $t$ and remained loyal then experimentation does not occur in period $t+1$. A consumer's information is then no more than one period old. For example, if $x_1^{t-1} > x_2^t$ then type 0 customers who are loyal to firm 1 and buy from firm 2 switch loyalty to firm 2. Thus, firm 1 only retains $1-\rho$ of its $\beta^t$ type 0 consumers who were loyal to it in the previous period. If, in addition, $x_1^t < x_2^{t-1}$ then type 0 customers who are loyal to firm 2 and buy from firm 1 will switch loyalty to firm 1. There are $\rho(\alpha - \beta^t)$ such consumers. As a result, when $x_1^{t-1} > x_2^t$ and $x_1^t < x_2^{t-1}$, $\beta^{t+1} = (1-\rho)\beta^t + \rho(\alpha - \beta^t)$. The full equations of motion are provided in Appendix A.

These dual dynamics create a feedback system defined on the state variables, $(x_1^{t-1}, x_2^{t-1}, \beta^t, \gamma^t)$. The dynamic on firm practices in (3) depends on the current allocation of loyal customers across firms while the allocation of customer loyalty depends on firms' attributes. This feedback mechanism has the potential for creating market dominance.

For suppose firm 1's share of loyal customers is biased towards type 0 consumers. Then a new idea that generates more profit out of type 0 consumers will generate greater virtual profit to firm 1. It is then inclined to adopt such an idea and, by the same reasoning, is disinclined to adopt ideas appealing to type 1 consumers. Furthermore, if firm 1's loyal customers are biased to being type 0 then this must mean that firm 2's loyal customers tend to be of type 1. By an analogous logic, firm 2 is inclined to adopt ideas that appeal to type 1 consumers. In this manner, firm 1 will become increasingly attractive (relative to firm 2) to type 0 consumers and thus induce more of them to become loyal to it. This is the potential for increasing returns as an initial stock of loyal consumers biased to being type 0 can induce adoption of ideas by a firm that causes the new stock of loyal consumers to be even more heavily type 0. Given such a feedback mechanism, one would expect there to be events in which one firm is increasingly dominant. The real issue is whether that is necessarily the case and, if it can occur, whether it can lead to one firm permanently dominating the firm or, because firms are continually innovating, must there eventually be a disruption of the current market structure.[11]

While this particular feedback system between firms and consumers is new, previous work has modelled the dynamic movement of buyers among sellers. In Bergemann and Välimäki (1997), a new firm's product is of unknown quality and both buyers and sellers receives signals; the informativeness of which is increasing in the number of units sold of the product. Buyers move among sellers as they learn about the new seller's quality. In the search model of Burdett and Coles (1997), consumers know the price distribution in the market but not the price that each firm charges. Consumers enter the market and engage in costly price search (products are homogeneous). In each period, a firm has a stock of regular customers who are defined to be those that bought from it last period. They avoid search costs by buying from the firm again. This gives a firm some market power over its regular customers which creates feedback as a firm's stock of regular customers influences its price which determines next period's stock of regular customers. Weisbuch et al. (2000) explore the extent to which buyers and sellers form long-lasting relationships. In each period, buyers decide which seller to visit using reinforcement learning with the probability of visiting a seller depending on the past profit realized by interacting with that seller. Finally, Currie and Metcalfe (2001) consider competing duopolists who use heuristics to choose price, production, and investment, while consumers determine loyalty on the basis of price though subject to some inertia in their switching behavior. One of the main objectives of their analysis is to characterize those situations for which a less efficient firm is driven out of the industry.

[11] It is worth noting that the random element in the model are firms' practices rather than consumers' loyalty decisions. This choice is motivated by our sense that while there may be some randomness in an individual consumer's loyalty decision, the law of large numbers would tend to operate at the level of a firm's customers. That is, it is unlikely that a firm would experience a substantive change in its customer base due to random actions by consumers. In contrast, we believe that innovation is highly stochastic; there is a fair amount of randomness associated with coming up with new ideas.

## 3. Nash equilibrium

Prior to analyzing adaptive dynamics, it is useful to characterize Nash equilibrium for the complete information static game as a benchmark. Imagine that firms know the distribution of consumers, the rate at which consumers buy from them, and the space of attributes. Thus, contrary to the preceding model, firms know all that could be known about how to satisfy consumers. A firm is modelled as choosing attributes to maximize its profit given the (correctly) anticipated attributes of the other firm's product and the (correctly) anticipated sorting by consumers. Firm $i$'s payoff is then [12]

$$\pi_i = \begin{cases} (1-\rho)\alpha g(x_i) + \rho(1-\alpha)g(1-x_i) & \text{if } x_i < x_j, \\ (\alpha/2)g(x_i) + [(1-\alpha)/2]g(1-x_i) & \text{if } x_i = x_j, \\ \rho\alpha g(x_i) + (1-\rho)(1-\alpha)g(1-x_i) & \text{if } x_j < x_i. \end{cases} \tag{5}$$

By locating to the left (right) of its competitor, a firm induces all type 0 (1) consumers to be loyal to it. If it locates exactly at the other firm's location then the two firms equally divide the set of consumers.

As results in this section are defined for when $X$ is sufficiently dense, denumerate the elements of $X$ so that $X = \{x(0), x(1), \ldots, x(K)\}$ where $x(0) = 0 < x(1) < \cdots < x(K-1) < x(K) = 1$. Define $\varepsilon(X) \equiv \max\{x(h+1) - x(h)\colon h \in \{0, 1, 2, \ldots, K-1\}\}$ as the maximal distance between adjacent prices. Theorem 2 shows that, when $X$ is sufficiently dense, if the proportion of type 0 consumers is sufficiently high then an equilibrium exists. Furthermore, the equilibrium is unique and has both firms deploying the ideal practice for type 0 consumers and thereby sharing the market. [13]

**Theorem 2.** $\exists \bar{\varepsilon} > 0$ *such that if* $\varepsilon(X) \in (0, \bar{\varepsilon})$ *then* $\exists \underline{\alpha} \in (\frac{1}{2}, 1)$ *such that*: (i) *if* $\alpha \in (\frac{1}{2}, \underline{\alpha})$ *then a pure-strategy Nash equilibrium does not exist*; *and* (ii) *if* $\alpha \in [\underline{\alpha}, 1]$ *then* $(x_1, x_2) = (0, 0)$ *is the unique pure-strategy Nash equilibrium.*

If firms have different locations, say $x_1 < x_2$, then firm 1 is attracting type 0 consumers and firm 2 is attracting type 1 consumers. Firm 2 can then improve its profit by locating just to the left of $x_1$ and attracting type 0 consumers because there are more of them than type 1 consumers. The only way that cannot happen is if $x_1 = 0$. However, if $\alpha$ is sufficiently close to $\frac{1}{2}$ then firm 2 prefers to locate at $\bar{\phi}$ (the optimal location when all of its loyal customers are type 1) and focus on serving type 1 consumers than to locate at 0 and share both consumer types. But if it does that then firm 1 prefers to locate at $\underline{\phi}$ (the optimal location when all of its loyal customers are type 0). Hence, an equilibrium does not exist when $\alpha$ is low. When $\alpha$ is sufficiently high, both firms are content to locate at 0 and share the market rather than be the exclusive preferred provider for the minority consumer type.

[12] To make an appropriate comparison with the dynamic model, $\rho$ is maintained and not set equal to zero though, as we will see, qualitative results are independent of $\rho$.

[13] The proof of Theorem 2 also shows that the result extends to when $X = [0, 1]$.

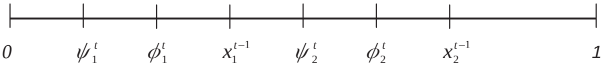


Fig. 2.

## 4. Sustained market dominance

A state in the system is a pair of locations and an allocation of customers to firms in terms of their loyalty: $(x_1, x_2, \beta, \gamma)$. While $(x_1, x_2)$ lies in the finite space $X$, $(\beta, \gamma)$ lies in the continuum, $[0, \alpha] \times [0, 1-\alpha]$. Although the state space is then uncountable, additional structure will allow us to use the theory of finite Markov chains. In particular, note that the only randomness in $(\beta, \gamma)$ is from randomness in $(x_1, x_2)$. Furthermore, if the ordering between $(x_1, x_2)$ does not change then $(\beta, \gamma)$ evolves deterministically though, for generic initial conditions, only settles down in the limit. [14]

We will use the term *dominance* to refer to one firm having more than half of the market which will often mean having almost all type 0 consumers as loyal customers. Our primary interest is in characterizing long-run states and determining whether they are characterized by dominance. For this purpose, we define an absorbing state to be one that persists over time.

**Definition 1.** $(\hat{x}_1, \hat{x}_2, \hat{\beta}, \hat{\gamma})$ is an absorbing state if $(x_1^{t-1}, x_2^{t-1}, \beta^t, \gamma^t) = (\hat{x}_1, \hat{x}_2, \hat{\beta}, \hat{\gamma})$ implies $(x_1^t, x_2^t, \beta^{t+1}, \gamma^{t+1}) = (\hat{x}_1, \hat{x}_2, \hat{\beta}, \hat{\gamma})$ with probability one.

Theorem 3 shows that there are three absorbing states. Two of these have a dominant firm – one has firm 1 capturing all type 0 consumers and the other has firm 2 capturing them – and the third has firms equally sharing the market.

**Theorem 3.** *The set of absorbing states is* $\{(\underline{\phi}, \bar{\phi}, \alpha, 0), (\bar{\phi}, \underline{\phi}, 0, 1-\alpha), (\hat{\phi}, \hat{\phi}, \alpha/2, (1-\alpha)/2)\}$.

The remainder of the section explores the extent to which dynamics lead the system to these absorbing states. The first point to note is that given the equations of motion for $(\beta^t, \gamma^t)$, customer allocations will never reach their values at an absorbing state except for the non-generic event that they start at those values. For example, even when $(x_1^t, x_2^t) = (\underline{\phi}, \bar{\phi})\,\forall t$, if $(\beta^1, \gamma^1) \neq (\alpha, 0)$ then, since $\beta^{t+1} = \beta^t + \rho(\alpha - \beta^t)$, $(\beta^t, \gamma^t) \neq (\alpha, 0)\,\forall t$ though $\lim_{t\to\infty}(\beta^t, \gamma^t) = (\alpha, 0)$. Therefore, at best, we can expect the system to converge to an absorbing state though never actually be in an absorbing state.

As an initial step, we characterize a set of states such that the asymmetric absorbing states are reached in the limit with probability one. As defined below, $\Omega_i$ is the set of states such that firms' sets of acceptable ideas do not intersect and firm $i$'s maximal acceptable idea is less than firm $j$'s minimal acceptable idea. See Fig. 2 for an example

[14] For example, if $x_1^{t-1} < x_2^t$ and $x_1^t < x_2^{t-1}\,\forall t \geqslant t'$ then $\beta^{t+1} = \beta^t + \rho(\alpha - \beta^t)$ and $\gamma^{t+1} = (1-\rho)\gamma^t\,\forall t \geqslant t'$.

of a state in $\Omega_1$.

$$\Omega_i \equiv \{(x_1, x_2, \beta, \gamma) : \max\{\psi_i(x_i, \beta, \gamma), x_i\} < \min\{\psi_j(x_j, \beta, \gamma), x_j\}, j \neq i\} \subseteq [0,1]^2 \times [0, \alpha] \times [0, 1-\alpha]. \tag{6}$$

Next define

$$\bar{\Omega}_i \equiv \Omega_i \cap X^2 \times [0, \alpha] \times [0, 1-\alpha]$$

as the subset of $\Omega_i$ that includes only feasible locations.

Theorem 4 shows that if the state is in $\bar{\Omega}_i$ then firm $i$ dominates for sure in that the system converges to the asymmetric absorbing state in which firm $i$'s loyal customer base is comprised of all type 0 consumers. Once in $\bar{\Omega}_1$ or $\bar{\Omega}_2$, the dynamic path on market shares is deterministic with the dominant firm steadily attracting more type 0 consumers and steadily losing type 1 consumers. However, the path on firms' attributes is stochastic and, furthermore, as shown in the proof, each firm's attributes will generally not be monotonic. Also, note that $(\underline{\phi}, \bar{\phi}, \alpha, 0) \in \bar{\Omega}_1$ and that any state sufficiently close to $(\underline{\phi}, \bar{\phi}, \alpha, 0)$ is also in $\bar{\Omega}_1$. Thus, if the state is near $(\underline{\phi}, \bar{\phi}, \alpha, 0)$ then it converges almost surely to $(\underline{\phi}, \bar{\phi}, \alpha, 0)$. An analogous statement applies to $(\bar{\phi}, \underline{\phi}, 0, 1-\alpha)$ and $\bar{\Omega}_2$. In this sense, the asymmetric absorbing states are locally stable.

**Theorem 4.** *If* $(x_1^0, x_2^0, \beta^1, \gamma^1) \in \bar{\Omega}_1$ *then, with probability one,*

$$\lim_{t\to\infty} (x_1^{t-1}, x_2^{t-1}, \beta^t, \gamma^t) = (\underline{\phi}, \bar{\phi}, \alpha, 0). \tag{7}$$

*If* $(x_1^0, x_2^0, \beta^1, \gamma^1) \in \bar{\Omega}_2$ *then, with probability one,*

$$\lim_{t\to\infty} (x_1^{t-1}, x_2^{t-1}, \beta^t, \gamma^t) = (\bar{\phi}, \underline{\phi}, 0, 1-\alpha). \tag{8}$$

If the state is in $\bar{\Omega}_1$ then it implies that $x_1^{t-1} < x_2^{t-1}$ so that type 0 consumers prefer firm 1's product. If this ordering of firms' attributes persists then, due to continual consumer experimentation, all type 0 consumers will eventually learn that firm 1 better meets their needs and thus become loyal to firm 1. Similarly, type 1 consumers will eventually all be loyal to firm 2. The next issue is what ensures that this ordering of firms' attributes persists. Note that firm 1 does not adopt any idea in period $t$ which exceeds $\max\{\psi_1(x_1^{t-1}, \beta^t, \gamma^t), x_1^{t-1}\}$ and firm 2 does not adopt any idea which is less than $\min\{\psi(x_2^{t-1}, \beta^t, \gamma^t), x_2^{t-1}\}$. Since $\max\{\psi_1(x_1^{t-1}, \beta^t, \gamma^t), x_1^{t-1}\} < \min\{\psi(x_2^{t-1}, \beta^t, \gamma^t), x_2^{t-1}\}$ then $x_1^t < x_2^t$ so that this ordering is sure to continue into the next period. This is not sufficient to ensure the result, however, because $\psi_1(x_1^{t-1}, \beta^t, \gamma^t)$ is not monotonically decreasing over time and $\psi_2(x_1^{t-1}, \beta^t, \gamma^t)$ is not monotonically increasing over time and, therefore, firms' locations are not monotonic over time even when the state lies in $\bar{\Omega}_1$. However, the proof of Theorem 4 shows that $\max\{\psi_1(x_1^{t-1}, \beta^t, \gamma^t), x_1^{t-1}\}$ is monotonically decreasing over time and $\min\{\psi(x_2^{t-1}, \beta^t, \gamma^t), x_2^{t-1}\}$ is monotonically increasing over time. As firm 1 acquires more type 0 consumers and fewer type 1 consumers as loyal customers, its set of acceptable and feasible ideas shifts to the left and firm 2's set shifts to the right. Thus, if they do not intersect initially then they do not intersect in any future period.[15] As a result, $x_1^\tau < x_2^\tau \, \forall \tau \geqslant t$ and therefore firm 1 will eventually have all type 0 consumers loyal to it.

While the asymmetric absorbing states are locally stable, to what extent can the system reach them globally? And to what extent is the system drawn to the symmetric absorbing state? The next result is relevant to addressing both questions.

**Theorem 5.** *If $x_1^0 \neq x_2^0$ then, with positive probability,*

$$\lim_{t\to\infty} (x_1^{t-1}, x_2^{t-1}, \beta^t, \gamma^t) \in \{(\underline{\phi}, \bar{\phi}, \alpha, 0), (\bar{\phi}, \underline{\phi}, 0, 1-\alpha)\}.$$

Note that Theorem 5 implies that the symmetric state is not locally stable in the sense that when firms' locations are different, even if they are close to $(\hat{\phi}, \hat{\phi})$, the system will converge to an asymmetric absorbing state with positive probability. This result is independent of how dense $X$ is so that $(x_1^{t-1}, x_2^{t-1})$ could be arbitrarily close to $(\hat{\phi}, \hat{\phi})$ and firms still might not return to $(\hat{\phi}, \hat{\phi})$. The second implication to note is that the asymmetric states can be reached with positive probability from any initial state as long as firms' locations are distinct. If firms' locations are distinct, it is then possible that those locations will persist for a sufficiently long time that most type 0 consumers will be loyal to one firm and most type 1 consumers will be loyal to the other firm. From that point, firms will only adopt locations that will reinforce the bias in their customer base. They are then on a path that leads to an asymmetric absorbing state for sure. We conclude that an asymmetric situation is, speaking imprecisely but still meaningfully, a more robust attractor than the symmetric state.

As described earlier, implicit in our model is an increasing returns mechanism. A firm that currently has a customer mix biased toward the prevalent consumer type in the market will tend to identify as valuable those ideas well suited to that type. Their adoption impacts future loyalty switching by consumers and generally leads to a customer mix even more biased toward the prevalent type which makes the firm more inclined to adopt ideas suitable for them. Eventually, this process results in one of the firms capturing and retaining most of the market. What remains to be explained is why it is absorbing. Indeed, with positive probability, a market laggard (one that is catering to type 1 consumers) will come up with an idea that could attract type 0 consumers away from the market leader. Indeed, any location between 0 and that of the market leader's location will suffice. The problem is that the market laggard rejects such an idea because it is concerned with its own customer base. Thus, the absorbing nature of market dominance – and why it can be permanently sustained – is that a firm's future path is necessarily constrained by its desire to please its current customers. This is exactly the type of bias that was highlighted in the analysis of Christensen (1997). While the result is generated with a highly simplified model, the underlying story seems quite general.

In Harrington and Chang (2001), the continuum case is examined as firms' locations lie in [0,1]. Qualitatively similar but stronger results are derived though with more complex proofs. It is shown that, almost surely, the system converges to one of the

[15] This property does require that the profit functions are well behaved and, in particular, that $g$ is concave.

asymmetric absorbing states. Thus, sustained market dominance always prevail in that case.

## 5. Comparison of the adaptive dynamic with Nash equilibrium

In summarizing the results of the previous two sections, a classical equilibrium analysis generates very different predictions than our model of firm and consumer adaptation. Nash equilibrium produces a symmetric outcome with firms locating at 0 so as to best satisfy the most prevalent consumer type. By contrast, adaptive dynamics always result in firms locating in the interior and can produce either market dominance – with firms locating at $(\underline{\phi}, \bar{\phi})$ or $(\bar{\phi}, \underline{\phi})$ – or a symmetric outcome with both firms locating at $\hat{\phi}$. The objective of this section is to explain the disparity in these results.

A crucial distinction in these two models is whether a firm perceives its customer base as exogenous or endogenous. Implicit in the Nash equilibrium description of behavior is that a firm takes as fixed the other firm's location but expects consumers to fully respond by going to the firm with the best location. For example, $(\underline{\phi}, \bar{\phi})$ is not a Nash equilibrium of the game of Section 3 because, by locating to the left of $\underline{\phi}$, firm 2 anticipates attracting all type 0 consumers while those consumers would go to firm 1 if firm 2 located at $\bar{\phi}$. Such a response by type 0 consumers makes a move to $\underline{\phi}$ profitable for firm 2 which destabilizes $(\underline{\phi}, \bar{\phi})$. The ability to lure customers – an effect originally identified by Hotelling (1929) – induces each firm to move closer than its rival to the ideal location of the more numerous consumer type. Firms are then moving towards the same target and ultimately end up at 0.[16] In contrast, the adaptive dynamic can lead firms to move in opposite directions. Firms having different loyal customer bases will generate different target customer bases (by assumptions A4 – A5). If firm 1 has more type 0 consumers relative to firm 2 and firm 2 has more type 1 consumers relative to firm 1 then a location closer to 0 is valued more by firm 1 than by firm 2 and a location closer to 1 is valued more by firm 2 than by firm 1. This can result in firms moving in different directions – firm 1 towards 0 and firm 2 towards 1 – and result in an asymmetric outcome being an absorbing state. Critical to this argument is that consumers are not fully and instantaneously adjusting their loyalties to firms' locations. With partial adjustment, a firm's current customer base matters and that is what leads firms to attach different evaluations to the same idea. In other words, firms are climbing different landscapes by virtue of how their current loyal customers influences that landscape. In contrast, the full and instantaneous consumer adjustment under a classical game-theoretic approach makes a firm's current customers irrelevant so firms are climbing the same landscape which ultimately leads to symmetry in their final locations and thereby the absence of market dominance.

[16] The stability of firms at (0,0) does require that $\alpha$ be sufficiently close to 1 (see Theorem 2). If $\alpha$ is close to $\frac{1}{2}$ then the resulting dynamic story is instead similar to the Edgeworth cycle. Firms move closer to 0 but once a firm is close enough, it moves closer to 1. Such a move results in it conceding type 0 consumers to the other firm and locating so as to generate more profit from the type 1 consumers that it attracts.

A second approach to explaining these different outcomes is to expand the class of adaptive dynamics so that, under some conditions, the Nash equilibrium outcome is an absorbing state. One can then compare the properties of the adaptive dynamics that result in that outcome as opposed to the absorbing states of Theorem 3. Towards that end, modify the original model by assuming that a firm adopts a new location when it generates average profit over the next $T$ periods that exceeds the average profit (over the next $T$ periods) from its existing location, assuming firms' locations remain fixed thereafter and consumers engage in partial adjustment as specified in Section 2. The adaptive dynamic explored in the previous section is the case of $T=1$ as a firm is myopic in only considering current profit. Thus, when $T=1$ the set of absorbing location pairs for this dynamic is $\{(\underline{\phi},\bar{\phi}),(\bar{\phi},\underline{\phi}),(\hat{\phi},\hat{\phi})\}$. We will argue next that if $T$ is sufficiently large then the Nash equilibrium outcome is an absorbing state of this dynamic.

First note that the average profit for a firm at the state $(0,0,\alpha/2,(1-\alpha)/2)$ is $(\alpha/2)g(0)+((1-\alpha)/2)g(1)$. Now consider, say, firm 1 locating at $x'>0$ so that type 0 consumers prefer firm 2 and type 1 consumers prefer firm 1. As a result, a fraction $\rho$ of firm 1's loyal type 0 consumers will switch loyalties to firm 2 each period (as that is the fraction that is searching) and a fraction $\rho$ of firm 2's type 1 consumers will switch to firm 1. Starting with $(\beta^0,\gamma^0)=(\alpha/2,(1-\alpha)/2)$, firm 1's loyal customer base in $t$ periods is

$$\beta^t=(1-\rho)^t(\alpha/2)$$

$$\begin{aligned}\gamma^t&=((1-\alpha)/2)+[1-(1-\rho)^t]((1-\alpha)/2)\\&=(1-\alpha)-(1-\rho)^t((1-\alpha)/2).\end{aligned}$$

It follows that firm 1's profit in $t$ periods is

$$\begin{aligned}\pi_1^t=&[(1-\rho)(1-\rho)^t(\alpha/2)+\rho(\alpha-(1-\rho)^t(\alpha/2))]g(x')\\&+[(1-\rho)((1-\alpha)-(1-\rho)^t((1-\alpha)/2))\\&+\rho(1-\rho)^t((1-\alpha)/2)]g(1-x'),\end{aligned}$$

where recall that a firm's consumers are comprised of $1-\rho$ of its loyal customers and $\rho$ of the other firm's loyal customers. Since

$$\lim_{t\to\infty}\pi_1^t=\rho\alpha g(x')+(1-\rho)(1-\alpha)g(1-x')$$

then

$$\lim_{T\to\infty}\sum_{t=0}^{T}\left(\frac{\pi_1^t}{T}\right)=\rho\alpha g(x')+(1-\rho)(1-\alpha)g(1-x').$$

This is the exact same profit as for the game-theoretic model (see (5)) and, by Theorem 2, we know that when $\alpha$ is sufficiently high that this profit is less than that from locating at 0. We conclude that if firms evaluate ideas based on the long-run average profit then the Nash equilibrium is an absorbing state. To summarize, in the face of gradual consumer adjustment, adaptive dynamics can yield asymmetric outcomes and market dominance when firms are myopic while the Nash equilibrium outcome emerges when firms are far-sighted and infinitely patient.[17]

Before moving on, we would like to make two final remarks in defense of the adaptive dynamic that generates market dominance. First, though these asymmetric absorbing states are not Nash equilibria, they are local Nash equilibria as each firm's location is locally optimal. Recall that $\underline{\phi}$ is optimal for firm 1 when all type 0 consumers are loyal to it and $\bar{\phi}$ is optimal to firm 2 when all type 1 consumers are loyal to it. Furthermore, this customer allocation is stable at $(\underline{\phi}, \bar{\phi})$ – as type 0(1) consumers prefer firm 1(2) – and, most critically, in a neighborhood of it. Thus, a firm's beliefs that the customer allocation will not change in response to its location actually proves to be right when the location is nearby. Firms are then locally optimizing and $(\underline{\phi}, \bar{\phi})$ is a local Nash equilibrium. Second, while a Nash equilibrium does not always exist, the adaptive dynamic always has an absorbing state.

## 6. Comparative dynamics and simulations

To explore the presence of a first-mover advantage and its determinants, simulations were conducted. A *first-mover advantage* will refer to any advantage emanating from the initial conditions to the system. Assume that a firm's target customer base is their current customer base, as specified in (4): a firm adopts a new idea if and only if it raises current profit. The system then has three parameters: $\alpha$ (the proportion of type 0 consumers in the population), $\rho$ (the rate at which consumers experiment), and $\omega$ (the rate at which firms discovers new ideas); and four initial conditions: $\beta^1$ (the initial proportion of type 0 consumers loyal to firm 1), $\gamma^1$ (the initial proportion of type 1 consumers loyal to firm 1), $x_1^0$ (the initial location of firm 1), and $x_2^0$ (the initial location of firm 2). It is further assumed that $g(|k-x|) = 1-(k-x)^2$. The ensuing long-run locations are

$$\underline{\phi} = \frac{\rho(1-\alpha)}{(1-\rho)\alpha + \rho(1-\alpha)}, \quad \bar{\phi} = \frac{(1-\rho)(1-\alpha)}{(1-\rho)(1-\alpha)+\rho\alpha}.$$

Simulations involve a four step procedure. First, values are set for $\alpha, \rho, \omega, \beta^1$, and $\gamma^1$. Second, values for $x_1^0$ and $x_2^0$ are randomly selected from $X$ according to a uniform distribution. $X$ is set at the computer's representation of $[0,1]$. In that it is then very unlikely for firms to have identical locations, the symmetric absorbing state is reached with very low probability. The purpose of the simulations is instead to explore what factors are conducive to a specific firm dominating. Third, the model is played out which involves generating a sequence of ideas and having firms and consumers respond to that sequence according to the equations of motion. The second and third steps are repeated 1000 times. The values reported are the averages of these 1000 runs.

The height of the surface in Fig. 3 measures the frequency with which firm 1 dominates so that the long-run market share of firm 1 is $(1-\rho)\alpha + \rho(1-\alpha)$. Its dependence on firm 1's initial share of type 0 consumers, $\beta^1/\alpha$, and its initial share of

[17] We would like to thank a referee for suggesting this line of explanation.

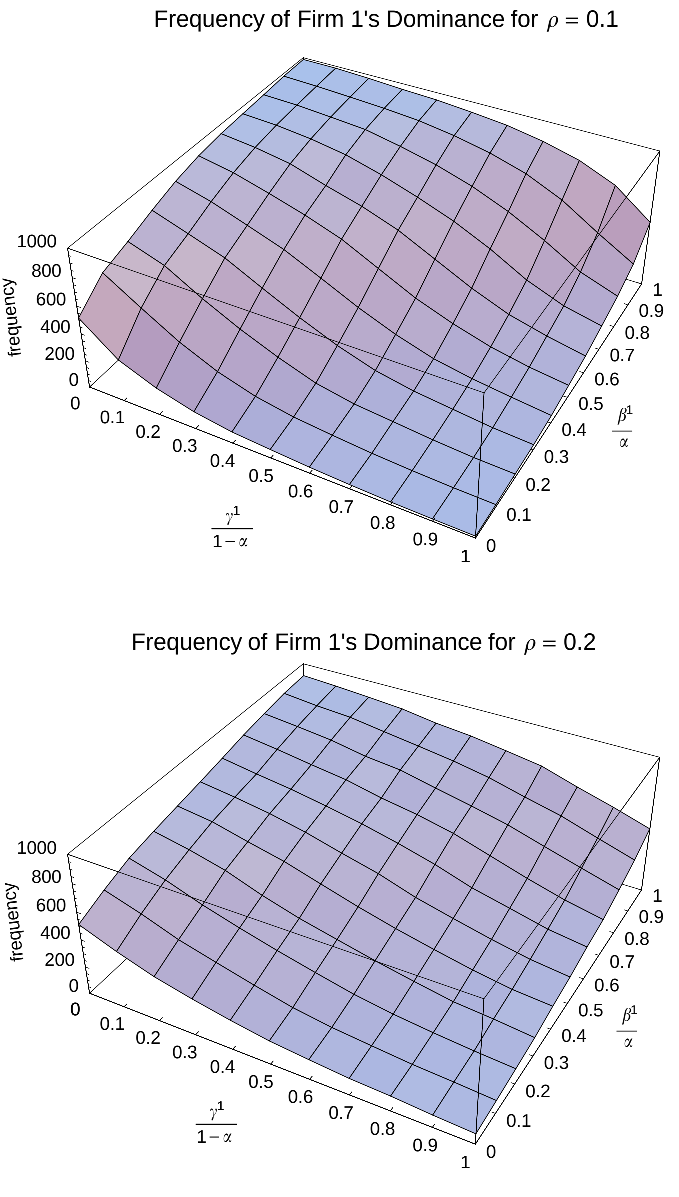


Fig. 3. $\alpha = 0.6$, $\omega = 0.7$.

type 1 consumers, $\gamma^1/(1-\alpha)$, is shown. These results are for when 60% of consumers are type 0 ($\alpha = 0.6$), on average a firm receives seven ideas every ten periods ($\omega = 0.7$), and on average a consumer experiments once every ten periods ($\rho = 0.1$) and once every five periods ($\rho = 0.2$). Fig. 3 shows that a higher mass of loyal type 0 consumers

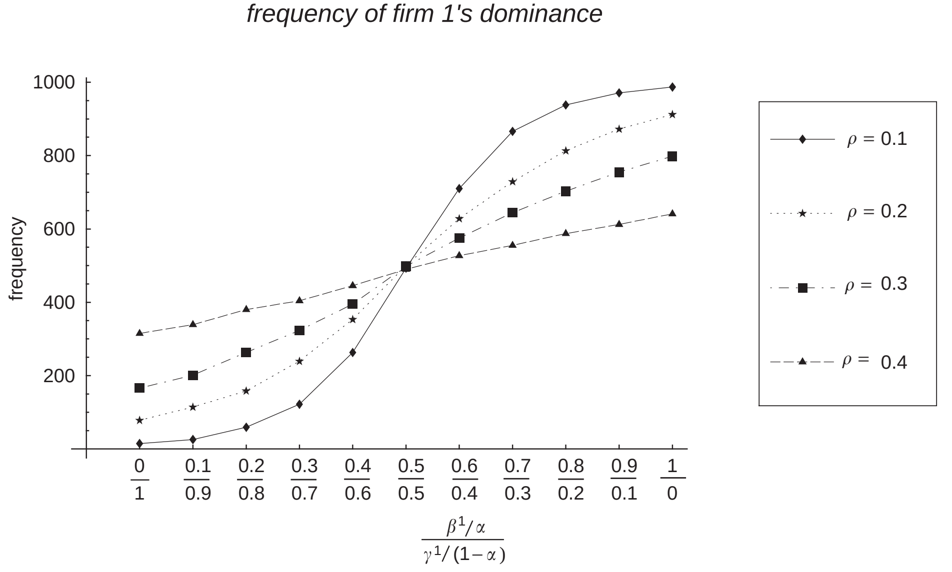


Fig. 4. Impact of $\rho$ ($\alpha = 0.6$; $\omega = 0.7$).

and a lower mass of loyal type 1 consumers increases the frequency with which firm 1 dominates. By having an initial customer mix biased towards type 0 consumers, firm 1 is more inclined to adopt ideas suitable for type 0 consumers and this ultimately enhances the likelihood of dominating the market.

A second question to explore with simulations is to what extent the rate of consumer experimentation is complementary to this first-mover advantage. There are two countervailing forces at play. If, say, firm 1 has a higher mix of type 0 consumers, it is more likely than the other firm to adopt a location that is more attractive to those consumers. If consumers experiment at a higher rate, type 0 consumers who are currently loyal to the other store will then learn about the firm's superior product and flow to it quicker. This makes it more likely that the state will get into $\bar{\Omega}_1$. By this argument, a higher value for $\rho$ augments the first-mover advantage from a higher mix of type 0 consumers. On the other hand, if the current market laggard, in terms of the customer mix, is able to develop a superior product then more consumer experimentation will result in a heavier flow of type 0 consumers to it. It may then be able to become a market leader before the other firm develops a yet even better practice. In other words, a higher rate of consumer experimentation can allow the market leader to more quickly capitalize on its lead but can also allow a market laggard to more quickly supplant the current leader. Examination of Fig. 3 suggests that the latter effect dominates. As $\rho$ is increased from 0.1 to 0.2, the relationship between initial customer mixes and the frequency with which firm 1 dominates becomes flatter; meaning that the likelihood of dominance is less responsive to a firm's customer base. Fig. 4 shows this more generally. The horizontal axis measures the degree of firm 1's first-mover advantage where it has no advantage at 0.5/0.5 and, from that point upward, its advantage is increasing. As $\rho$ increases, the curve flattens which indicates that, for any initial advantage, the frequency with which firm 1 dominates is reduced. From these results, it is concluded that a higher rate of consumer experimentation weakens a first-mover advantage.

## 7. Horizontally and vertically differentiated practices

While the previous model showed how sustained market dominance can prevail, it had little to say about when we would expect to observe it since such an outcome is always an absorbing state. Towards addressing that question, we now enrich the model to allow firms' products to be both horizontally and vertically differentiated; that is, the quality of their products can differ.

Let $z_i^t$ denote the quality of firm $i$'s product in period $t$ and assume it can take on one of a finite number of values from $[0, \bar{z}]$ where $\bar{z} > 0$. A type $k$ consumer prefers a firm with attributes $x'$ and quality $z'$ to a firm with $x''$ and $z''$ iff $|x' - k| - |x'' - k| < z' - z''$. If one firm's product is both closer to a firm's ideal set of attributes and is of higher quality then clearly it is preferred. When, for example, $x' < x''$ and $z' < z''$ then a type 0 consumer prefers product $(x', z')$ iff the gain in the horizontal dimension, $x'' - x'$, exceeds the loss in the vertical dimension, $z'' - z'$. Given these preferences, it is straightforward to adapt the equations of motion on customer loyalties. Next assume that quality affects current (and virtual) profit in a proportional manner so that we can retain $\tilde{\pi}_i(x_i^t, \beta^t, \gamma^t)$ as a firm's virtual profit function wlog.[18] Note that one can interpret firms as receiving ideas that affect the vertical dimension (that is, $z_i^t$) as well as the horizontal dimension (that is, $x_i^t$). Since profit is monotonically increasing in quality, it will always adopt quality-improving ideas.

Given our use of distance functions in consumer preferences, what is important for the analysis is not absolute quality but rather relative quality, $z^t \equiv z_1^t - z_2^t \cdot z^t \in \Delta$ where $\Delta$ is finite a subset of $[-\bar{z}, \bar{z}]$ . A state is now defined by

$$s^t \equiv (x_1^{t-1}, x_2^{t-1}, z^{t-1}, \beta^t, \gamma^t) \in \Gamma \equiv X^2 \times \Delta \times [0, \alpha] \times [0, 1-\alpha].$$

Define $\mu : \Delta \times \Gamma \to [0, 1]$ to be the probability function over $z^t$. For generality it is allowed to depend on the current state. Two assumptions are made on $\mu$. A6 requires that positive probability be assigned to firms having identical qualities and to the extreme values. A7 requires that the probability that the quality differential does not change over a finite number of periods is positive:

(A6) $\mu(z|s^t) > 0 \ \forall z \in \{-\bar{z}, 0, \bar{z}\}, \ \ \forall s^t \in \Gamma.$
(A7) If $\mu(z|s^t) > 0$ then $\forall$ finite $T$, $\prod_{\tau=t+1}^{t+T} \mu(z|s^\tau) > 0 \ \forall s^\tau$ such that $z^{\tau-1} = z$.

[18] Suppose profit is $h(z_i^t)\pi_i(x_i^t, \beta^t, \gamma^t)$ where $h(0) > 0$ and $h'(z_i^t) > 0$. If we then assume that virtual profit is $h(z_i^t)\tilde{\pi}_i(x_i^t, \beta^t, \gamma^t)$, the adoption decisions regarding new ideas are unaffected by the quality of practices.

To allow for more precise results, one final assumption is that a firm's optimal location depends only on the ratio of its mass of type 0 consumers to its mass of type 1 consumers.

(A8) $\phi$ is homogeneous of degree zero in $w(0)$ and $w(1)$.

A8 holds, for example, when virtual profit equals actual profit so that firms are engaging in myopic hill-climbing; as specified in (4).

As the quality differential is continually subject to random fluctuations, there does not exist an absorbing state in $\Gamma$. However, there will prove to be closed sets of states for which firms' locations do not change. We then define $(x_1, x_2)$ to be an absorbing pair of locations when there exists an initial customer allocation, $(\beta^1, \gamma^1)$, such that if $(x_1^0, x_2^0) = (x_1, x_2)$ then $(x_1^t, x_2^t) = (x_1, x_2)\,\forall t \geqslant 1$ for sure. This will require that firms' locations remain fixed given the future evolution of customer allocations and irrespective of the quality shocks.

**Definition 2.** $(x_1, x_2)$ is an absorbing pair of locations if $\exists \Lambda \subseteq [0, \alpha] \times [0, 1-\alpha]$ such that: if $s^t \in \{x_1\} \times \{x_2\} \times \Delta \times \Lambda$ then $s^{t+1} \in \{x_1\} \times \{x_2\} \times \Delta \times \Lambda$ with probability one.

Subject to one caveat to be mentioned after the theorem, Theorem 6 establishes that Theorem 3 is robust to allowing for vertical differentiation as long as the maximum quality differential is not too large.

**Theorem 6.** *If $\bar{z} \in [0, \bar{\phi} - \underline{\phi})$ then the set of absorbing pairs of locations is $\{(\underline{\phi}, \bar{\phi}), (\bar{\phi}, \underline{\phi}), (\hat{\phi}, \hat{\phi})\}$.*

When firms' qualities are identical, which is an event that can occur with positive probability for any finite length of time (by A7), firms' locations and customer bases evolve exactly as found in Section 4. Hence, the only candidates for absorbing location pairs are those described in Theorem 3. To begin, let us examine an asymmetric substate, $(\underline{\phi}, \bar{\phi}, \alpha, 0)$. Consider $\{(\alpha, 0)\}$ as a candidate for $\Lambda$ and suppose $(x_1, x_2, \beta, \gamma) = (\underline{\phi}, \bar{\phi}, \alpha, 0)$. Since the quality differential is bounded above by the difference in firms' locations, $\bar{z} < \bar{\phi} - \underline{\phi}$, type 0 consumers will prefer firm 1's location of $\underline{\phi}$ and type 1 consumers will prefer firm 2's location of $\bar{\phi}$ irrespective of their qualities. Thus, once the state is $(\underline{\phi}, \bar{\phi}, \alpha, 0)$, firms' locations and customers' loyalties remain fixed.

The same type of argument as used in the proof of Theorem 5 can establish that the asymmetric absorbing location pairs can be reached with positive probability when firms' locations are distinct. Although quality shocks do not alter market dominance being an attractor, it would seem that quality shocks can be expected to delay the time until sustained market dominance occurs. For example, suppose $0 < |z^t| < x_2^t - x_1^t < \bar{\phi} - \underline{\phi}$ so that firm 1 is attracting type 0 consumers and firm 2 type 1 consumers. The quality differential is sufficiently small that it does not impact consumers' loyalty decisions. In this case, firm 1 is on a path to sustained dominance. What a quality shock can do is to alter the flow of consumers. In particular, if firm 2 experiences a positive shock so that $z^{t+1} < 0$ and $x_2^{t+1} - x_1^{t+1} < |z^{t+1}|$ then both consumer types will choose to switch to firm 2. By disrupting the dynamic that is currently making one firm dominant, quality shocks can delay the time it takes until ultimately one firm has achieved a position of sustained dominance.

The presence of quality shocks mildly alters the result for the symmetric state $(\hat{\phi}, \hat{\phi}, \alpha/2, (1-\alpha)/2)$. As long as firms' qualities are identical, these locations and customer allocations will persist, by the logic used in proving Theorem 3. But consider what happens when, say, firm 1's product has a quality advantage. Given firms have identical locations, both consumer types will prefer firm 1 and this causes the customer allocation to move from $(\alpha/2, (1-\alpha)/2)$ towards $(\alpha, 1-\alpha)$. Thus, $\Lambda = \{(\alpha/2, (1-\alpha)/2)\}$ will not work to show that $(\hat{\phi}, \hat{\phi})$ is an absorbing location pair. However, it is shown in the proof of Theorem 6 that if $(\beta^t/\gamma^t) = (\alpha/(1-\alpha))$ then $(\beta^{t+1}/\gamma^{t+1}) = (\alpha/(1-\alpha))$. Therefore, by A8, $\hat{\phi}$ remains each firm's optimal location so that $\Lambda = \{(\beta, \gamma) : (\beta/\gamma) = (\alpha/(1-\alpha))\}$ will work to show that $(\hat{\phi}, \hat{\phi})$ is an absorbing pair. Although dominance can switch between the two firms – according to the evolution of the quality differential – their locations remain at $(\hat{\phi}, \hat{\phi})$. The dominance of a particular firm is then temporary and vanishes when the other firm experiences a favorable quality differential for sufficiently long.

Now suppose the maximum quality differential is not constrained. Not surprisingly, Theorem 7 shows that there is no sustained dominance. Regardless of firms' current locations and customer bases, there will eventually be a sufficiently large quality shock that will shift the system to a path leading to the currently non-dominant firm becoming dominant.

**Theorem 7.** *If* $\bar{z} > \bar{\phi} - \underline{\phi}$ *then* $\forall s^t \in \Gamma$, $\exists$ *finite* $T$ *such that, with positive probability,* $\beta^{t+T} + \gamma^{t+T} > \frac{1}{2}$ *and, with positive probability,* $\beta^{t+T} + \gamma^{t+T} < \frac{1}{2}$.

Note that if $x_2 - x_1 > \bar{\phi} - \underline{\phi}$ then it is possible that no quality shock may be sufficient to alter the flow of type 0 consumers to firm 1 and type 1 consumers to firm 2 and thereby disrupt the growing dominance of firm 1. However, if this dynamic continues then, with positive probability in finite time, $(x_1, x_2) = (\underline{\phi}, \bar{\phi})$ at which point if the quality differential becomes less than $-(\bar{\phi} - \underline{\phi})$, both consumer types will flow to firm 2. In this manner, dominance can switch from firms 1 to 2 even though quality shocks are bounded below the maximum utility difference along the horizontal dimension.

To summarize, sustained market dominance can occur – with one firm's market share asymptotically approaching $(1-\rho)\alpha + \rho(1-\alpha)$ – when the maximum quality differential is less than $\bar{\phi} - \underline{\phi}$. When instead the maximum quality differential exceeds $\bar{\phi} - \underline{\phi}$, the identity of the market leader changes over time; there is no absorbing state with a particular firm being dominant. $\bar{\phi} - \underline{\phi}$ is then a critical value that determines whether or not sustained market dominance can emerge. To explore this issue, assume the specification in (4) so that the target customer base is the current customer base. Using the first-order conditions defining $\underline{\phi}$ and $\bar{\phi}$, it can be derived:

$$\frac{\partial \underline{\phi}}{\partial \rho} = \frac{\alpha g'(\underline{\phi}) + (1-\alpha) g'(1-\underline{\phi})}{(1-\rho)\alpha g''(\underline{\phi}) + \rho(1-\alpha) g''(1-\underline{\phi})} > 0,$$

$$\frac{\partial\bar{\phi}}{\partial\rho} = \frac{-\alpha g'(\bar{\phi}) - (1-\alpha)g'(1-\bar{\phi})}{\rho\alpha g''(\bar{\phi}) + (1-\rho)(1-\alpha)g''(1-\bar{\phi})} < 0,$$

so that $\bar{\phi} - \underline{\phi}$ is decreasing in $\rho$. Therefore, sustained market dominance is less likely when consumers experiment at a higher rate.

To understand this result, one must first recognize that the crucial issue regarding sustained market dominance is whether firms' locations can be sufficiently far apart in the long run so that even if the market laggard has higher quality, it does not alter consumers' loyalty decisions. When firms have comparable qualities and, say, firm 1 is dominant then firms' optimal attributes are stochastically converging to $\underline{\phi}$ for firm 1 and $\bar{\phi}$ for firm 2. Therefore, in finite time with positive probability, $|x_1^t - x_2^t| = |\bar{\phi} - \underline{\phi}|$. If $\bar{z} < \bar{\phi} - \underline{\phi}$, so that $\bar{z} < |x_1^t - x_2^t|$, then firm 2 cannot induce type 0 customers to become loyal to it even when it has higher-quality practices. This is the basis for Theorem 6. However, if $\bar{z} > \bar{\phi} - \underline{\phi}$ then higher quality induces type 0 customers to become loyal to firm 2 and, in fact, both consumer types are attracted to it. The role of the rate of consumer experimentation, $\rho$, is as follows. By raising $\rho$, firms have more similar mixes of consumers buying from them as, in any period, there is a larger fraction of 'noise' consumers; consumers who, in that period, are choosing a firm irrespective of their loyalty. The increased similarity in customer bases causes firms' long-run locations to be more similar. Hence, it becomes more likely that a quality advantage can cause consumers to switch loyalties and turn a market laggard into a market leader. In this manner, a higher rate of consumer experimentation makes sustained market dominance less likely.

## 8. Concluding remarks

If, as firms' locations settled down, consumer experimentation went to zero, it would not be surprising if sustained market dominance prevailed. Even if a non-dominant firm adopted a location that would be attractive to the prevalent consumer type, there would be little consumer response to it. If firms restricted themselves to discovering ideas close to their current location, it would also not be surprising if sustained dominance emerged. There might be ideas that would allow a non-dominant firm to become dominant but would never be found. Finally, if firms faced a cost to adjusting their location, it would once again not be surprising that sustained market dominance would emerge. What is striking about our analysis is that – in spite of consumers always engaging in experimentation, firms generating ideas from the entire space, and firms being able to costlessly adjust their locations – sustained market dominance can still prevail. Furthermore, this result is robust to the rate of consumer experimentation though it is not robust to allowing for sufficiently great shocks to the quality differential between firms' products.

In concluding, let us discuss two elements of our approach. First, price-setting behavior is not modelled. If prices are set to maximize static profit then allowing firms to choose prices should not upset our results. Since, in the absence of endogenizing prices, firms' attributes tend to diverge, allowing firms to choose price should reinforce that tendency since more similar products result in more intense price competition.[19] What would emerge if prices were set with some longer-run objective – for example, a firm charges a low price to lure consumers to try their product or service – is much less clear. Regardless, the model in its current form is relevant to those industries for which non-price competition is the primary avenue of competition. Some services are free to consumers – for example, network television and Internet portals and shopbots – as revenue is collected through advertising. Firms compete for consumers through product characteristics rather than price. In other industries, firms tacitly collude in price which redirects competition to other instruments at firms' disposal. An historically notable example is the U.S. cigarette industry. Industry observers noted that price was relatively fixed (at least until generic cigarettes were introduced) and firms competed in product traits, brand variety, and advertising.[20] More generally, in most industries competition occurs on multiple dimensions. If we feel we can learn something about firm behavior and market outcomes by focusing on price while excluding other instruments, there is reason to think that we can learn something by focusing on one of these other variables – product traits for the model at hand – while excluding price.

The other unique element of our model is characterizing firm and consumer behavior through the use of heuristics rather than equilibrium strategies.[21] Equilibrium is an assumption and, like all assumptions, must be judged on how compelling it is for the problem at hand. Is it reasonable for firms to have approximately accurate conjectures of competitors' and consumers' strategies and, given those beliefs, to have identified an approximately optimal solution? The firms in our model certainly have strong incentives to discover what is the best adoption rule concerning new ideas. Those decisions are directly relevant to the firm's profit and long-run survival. But the desire of a firm to determine optimal actions must be tempered by the complexity in figuring it out. We do believe that the environment of interest is of the level of complexity to warrant the exploration of non-equilibrium approaches. With that in mind, we have considered a wide class of heuristics plausibly consistent with how firms behave. Still, one could make other assumptions including the assumption of equilibrium. Until we have a clear idea of how firms make decisions in complex situations, the only safe recourse is to consider various approaches. On that note, we hope our analysis will inspire others to re-examine our setting under alternative behavioral assumptions.

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[19] However, the analysis of a Nash equilibrium in Section 3 would change if pricing decisions were modelled.

[20] This view is articulated in Tennant (1950) and Nicholls (1951).

[21] Other recent work utilizing a non-equilibrium dynamic approach to issues in oligopoly theory includes Luo (1995), Vega-Redondo (1997), Alós-Ferrer et al. (2000), Tanaka (2000), and Rhode and Stegeman (2001).

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## Appendix A.

The equation of motion on type 0 loyal customers is

$$\beta^{t+1} = \begin{cases} \beta^t + \rho(\alpha - \beta^t) & \text{if } x_1^{t-1} < x_2^t \text{ and } x_1^t < x_2^{t-1}, \\ \beta^t + (\rho/2)(\alpha - \beta^t) & \text{if } x_1^{t-1} < x_2^t \text{ and } x_1^t = x_2^{t-1}, \\ \beta^t & \text{if } x_1^{t-1} < x_2^t \text{ and } x_1^t > x_2^{t-1}, \\ (1 - (\rho/2))\beta^t + \rho(\alpha - \beta^t) & \text{if } x_1^{t-1} = x_2^t \text{ and } x_1^t < x_2^{t-1}, \\ (1 - (\rho/2))\beta^t + (\rho/2)(\alpha - \beta^t) & \text{if } x_1^{t-1} = x_2^t \text{ and } x_1^t = x_2^{t-1}, \\ (1 - (\rho/2))\beta^t & \text{if } x_1^{t-1} = x_2^t \text{ and } x_1^t > x_2^{t-1}, \\ (1 - \rho)\beta^t + \rho(\alpha - \beta^t) & \text{if } x_1^{t-1} > x_2^t \text{ and } x_1^t < x_2^{t-1}, \\ (1 - \rho)\beta^t + (\rho/2)(\alpha - \beta^t) & \text{if } x_1^{t-1} > x_2^t \text{ and } x_1^t = x_2^{t-1}, \\ (1 - \rho)\beta^t & \text{if } x_1^{t-1} > x_2^t \text{ and } x_1^t > x_2^{t-1}. \end{cases}$$

and on type 1 loyal customers is

$$\gamma^{t+1} = \begin{cases} (1 - \rho)\gamma^t & \text{if } x_1^{t-1} < x_2^t \text{ and } x_1^t < x_2^{t-1}, \\ (1 - \rho)\gamma^t + (\rho/2)(1 - \alpha - \gamma^t) & \text{if } x_1^{t-1} < x_2^t \text{ and } x_1^t = x_2^{t-1}, \\ (1 - \rho)\gamma^t + \rho(1 - \alpha - \gamma^t) & \text{if } x_1^{t-1} < x_2^t \text{ and } x_1^t > x_2^{t-1}, \\ (1 - (\rho/2))\gamma^t & \text{if } x_1^{t-1} = x_2^t \text{ and } x_1^t < x_2^{t-1}, \\ (1 - (\rho/2))\gamma^t + (\rho/2)(1 - \alpha - \gamma^t) & \text{if } x_1^{t-1} = x_2^t \text{ and } x_1^t = x_2^{t-1}, \\ (1 - (\rho/2))\gamma^t + \rho(1 - \alpha - \gamma^t) & \text{if } x_1^{t-1} = x_2^t \text{ and } x_1^t > x_2^{t-1}, \\ \gamma^t & \text{if } x_1^{t-1} > x_2^t \text{ and } x_1^t < x_2^{t-1}, \\ \gamma^t + (\rho/2)(1 - \alpha - \gamma^t) & \text{if } x_1^{t-1} > x_2^t \text{ and } x_1^t = x_2^{t-1}, \\ \gamma^t + \rho(1 - \alpha - \gamma^t) & \text{if } x_1^{t-1} > x_2^t \text{ and } x_1^t > x_2^{t-1}. \end{cases}$$

## Appendix B.

**Proof of Lemma 1.** Let us first show that if $\tilde{\pi}$ has an optimum, it is an interior solution. Consider $x = 0$:

$$\frac{\partial\tilde{\pi}(0, w(0), w(1))}{\partial x} = -\theta_1(w(0), w(1))g'(1) > 0$$

since $g'(0) = 0$. Hence, if $\phi$ exists then $\phi > 0$. Next consider

$$\frac{\partial\tilde{\pi}(1, w(0), w(1))}{\partial x} = \theta_0(w(0), w(1))g'(1) < 0. \tag{B.1}$$

Hence, if $\phi$ exists then $\phi < 1$. Given an optimum must be interior and $\tilde{\pi}$ is strictly concave then $\phi$ is defined by the first-order condition:

$$\theta_0(w(0), w(1))g'(\phi) - \theta_1(w(0), w(1))g'(1-\phi) = 0. \tag{B.2}$$

Define

$$\varDelta \equiv \theta_0(w(0), w(1))g''(\phi) + \theta_1(w(0), w(1))g''(1-\phi) < 0 \tag{B.3}$$

as $g$ is strictly concave. Taking the total derivative of (B.2) with respect to $w(0)$, one finds

$$\partial\phi/\partial w(0) = -[(\partial\theta_0/\partial w(0))g'(\phi) - (\partial\theta_1/\partial w(0))g'(1-\phi)]/\varDelta < 0 \tag{B.4}$$

since $\partial\theta_0/\partial w(0) > 0$, $\partial\theta_1/\partial w(0) \leqslant 0$, $g'(\phi) < 0$, and $g'(1-\phi) < 0$. Analogously,

$$\partial\phi/\partial w(1) = -[(\partial\theta_0/\partial w(1))g'(\phi) - (\partial\theta_1/\partial w(1))g'(1-\phi)]/\varDelta > 0. \qquad \square \tag{B.5}$$

**Proof of Theorem 2.** There are two possible outcomes: (i) $x_1 \neq x_2$ so that one firm's loyal customers are type 0 and the other firm's are type 1; and (ii) $x_1 = x_2$ so that each firm serves half of each consumer type. Let us first show that there does not exist an equilibrium with $x_1 \neq x_2$. Wlog, suppose $x_1 < x_2$ so that all type 0 consumers prefer to buy from firm 1 and all type 1 consumers prefer to buy from firm 2. It is immediate that $(x_1, x_2) = (\underline{\phi}, \bar{\phi})$ and firms' payoffs are

$$\pi_1^* \equiv (1-\rho)\alpha g(\underline{\phi}) + \rho(1-\alpha)g(1-\underline{\phi}),$$

$$\pi_2^* \equiv \rho\alpha g(\bar{\phi}) + (1-\rho)(1-\alpha)g(1-\bar{\phi}).$$

Let us first show that $\pi_1^* > \pi_2^*$. Suppose $\bar{\phi} > \frac{1}{2}$ and consider firm 1 locating at $1-\bar{\phi}$ and earning a payoff of

$$(1-\rho)\alpha g(1-\bar{\phi}) + \rho(1-\alpha)g(\bar{\phi}). \tag{B.6}$$

This payoff exceeds $\pi_2^*$ when

$$\begin{aligned}
&(1-\rho)\alpha g(1-\bar{\phi}) + \rho(1-\alpha)g(\bar{\phi}) > \rho\alpha g(\bar{\phi}) + (1-\rho)(1-\alpha)g(1-\bar{\phi}) \\
&\quad \Leftrightarrow \ (1-\rho)(2\alpha-1)g(1-\bar{\phi}) > \rho(2\alpha-1)g(\bar{\phi}) \\
&\quad \Leftrightarrow \ (1-\rho)g(1-\bar{\phi}) > \rho g(\bar{\phi}),
\end{aligned}$$

which is indeed true. Since $\pi_1^*$ is at least as great as the payoff in (B.6), we conclude $\pi_1^* > \pi_2^*$. Next suppose $\bar{\phi} \leqslant \frac{1}{2}$. As $\underline{\phi}$ is the unique optimal action when a firm's loyal customer base is all of the type 0 consumers then $\pi_1^*$ exceeds the payoff from locating at $\bar{\phi}$, holding fixed its loyal customer base to be all of the type 0 consumers:

$$(1-\rho)\alpha g(\underline{\phi}) + \rho(1-\alpha)g(1-\underline{\phi}) > (1-\rho)\alpha g(\bar{\phi}) + \rho(1-\alpha)g(1-\bar{\phi}). \tag{B.7}$$

Next note that the right-hand side of (B.7) exceeds $\pi_2^*$:

$$(1-\rho)\alpha g(\bar{\phi}) + \rho(1-\alpha)g(1-\bar{\phi}) > \rho\alpha g(\bar{\phi}) + (1-\rho)(1-\alpha)g(1-\bar{\phi})$$

$$\Leftrightarrow \quad (1-2\rho)\alpha g(\bar{\phi}) > (1-2\rho)(1-\alpha)g(1-\bar{\phi})$$

$$\Leftrightarrow \quad \alpha g(\bar{\phi}) > (1-\alpha)g(1-\bar{\phi}),$$

which is true since $\alpha > \frac{1}{2}$ and $\bar{\phi} \leqslant \frac{1}{2}$ implies $g(\bar{\phi}) \geqslant g(1-\bar{\phi})$. Therefore, $\pi_1^* > \pi_2^*$.

We are now prepared to prove that $(\underline{\phi}, \bar{\phi})$ is not a Nash equilibrium when $X$ is sufficiently dense. As $\underline{\phi} > 0$, firm 2 has the option of locating at $\underline{\phi} - \eta$, which is the location just below $\underline{\phi}$, and having all of the type 0 consumers as its loyal customer base. Since $\eta \leqslant \varepsilon(X)$, as $\varepsilon(X) \to 0$ then the profit to firm 2 from locating at $\underline{\phi} - \eta$ converges to $\pi_1^*$. Since $\pi_1^* > \pi_2^*$ – and these profit levels are independent of $\varepsilon(X)$ since $\underline{\phi}$ and $\bar{\phi}$ are independent of $\varepsilon(X)$ – then firm 2 prefers to locate at $\underline{\phi} - \eta$ than at $\bar{\phi}$ when $\varepsilon(X)$ is sufficiently small. Therefore, we conclude that when $X$ is sufficiently dense, there does not exist an asymmetric Nash equilibrium.

Now consider $(x_1, x_2) = (x, x)$. Each firm's payoff is $(\alpha/2)g(x) + [(1-\alpha)/2]g(1-x)$. Suppose $x \in X - \{0, 1\}$. A necessary condition for equilibrium is that locating at $x$ is preferable to locating at the next lowest value, denoted $x - \eta$, and having a loyal customer base of all of the type 0 consumers:

$$(\alpha/2)g(x) + [(1-\alpha)/2]g(1-x) \geqslant (1-\rho)\alpha g(x-\eta) + \rho(1-\alpha)g(1-x+\eta).$$

For this to hold $\forall \varepsilon(X) > 0$, it must hold $\forall \eta > 0$ which requires that

$$(\alpha/2)g(x) + [(1-\alpha)/2]g(1-x) \geqslant (1-\rho)\alpha g(x) + \rho(1-\alpha)g(1-x)$$

$$\Leftrightarrow \quad (1-\alpha)g(x) \geqslant \alpha g(1-x).$$

Another necessary condition is that locating at $x$ is preferable to locating at the next highest value, denoted $x + \eta$, and having a loyal customer base of all of the type 1 consumers:

$$(\alpha/2)g(x) + [(1-\alpha)/2]g(1-x) \geqslant \rho\alpha g(x+\eta) + (1-\rho)(1-\alpha)g(1-x-\eta).$$

For this to hold $\forall \varepsilon(X) > 0$, it must be true that

$$(\alpha/2)g(x) + [(1-\alpha)/2]g(1-x) \geqslant \rho\alpha g(x) + (1-\rho)(1-\alpha)g(1-x)$$

$$\Leftrightarrow \quad \alpha g(x) \geqslant (1-\alpha)g(1-x).$$

Combining these two conditions yields $\alpha g(x) = (1-\alpha)g(1-x)$. At a value of $x$ that satisfies that equality, a firm is indifferent between locating at $x$ and locating arbitrarily below $x$ (and focusing on type 0 consumers) and arbitrarily above $x$ (and focusing on type 1 consumers). If $\underline{\phi} < x$ then locating at $\underline{\phi}$ is strictly preferred to locating at $x - \eta$ as $\eta \to 0$. In that case, locating at $\underline{\phi}$ is strictly preferred to locating at $x$. It is then necessary that $x \leqslant \underline{\phi}$. By the same logic, it is necessary that $\bar{\phi} \leqslant x$. As this implies $x \leqslant \underline{\phi} < \bar{\phi} \leqslant x$, it is concluded that $\nexists x \in X - \{0,1\}$ such that $(x,x)$ is an equilibrium.

Consider $x = 1$. Each firm's payoff is $(\alpha/2)g(1) + [(1-\alpha)/2]g(0)$ and this is strictly less than locating at 0 and earning $(1-\rho)\alpha g(0) + \rho(1-\alpha)g(1)$. So (1,1) is not an equilibrium.

Finally, consider $x = 0$. The necessary and sufficient condition for equilibrium is that a firm prefers to locate at 0, and share both consumer types, than to locate at $\bar{\phi}$ and serve only type 1 consumers. This holds iff $\Psi(\alpha) \geqslant 0$ where

$$\Psi(\alpha) \equiv (\alpha/2)g(0) + ((1-\alpha)/2)g(1) - \rho\alpha g(\bar{\phi}) - (1-\rho)(1-\alpha)g(1-\bar{\phi}).$$

When $\alpha = \frac{1}{2}$, a firm strictly prefers to locate at 1 given the other firm is at 0 iff

$$(1-\rho)(\tfrac{1}{2})g(0) + \rho(\tfrac{1}{2})g(1) > (\tfrac{1}{4})g(0) + (\tfrac{1}{4})g(1),$$

which is indeed true. Hence, $\Psi(\frac{1}{2}) < 0$. Next note that it is an equilibrium for both firms to locate at 0 when $\alpha = 1$: $\Psi(1) = (\frac{1}{2})g(0) - \rho g(\bar{\phi}) > 0$. To conclude the proof, let us show that if $\Psi(\alpha) = 0$ then $\Psi'(\alpha) > 0$. By $\Psi$ being continuously differentiable, this implies that there is a unique value for $\alpha$ such that $\Psi(\alpha) = 0$.

$$\Psi'(\alpha) = (\tfrac{1}{2})[g(0) - g(1)] - [\rho g(\bar{\phi}) - (1-\rho)g(1-\bar{\phi})]$$
$$- \bar{\phi}'(\alpha)[\rho\alpha g'(\bar{\phi}) - (1-\rho)(1-\alpha)g'(1-\bar{\phi})].$$

Note that $\rho\alpha g'(\bar{\phi}) - (1-\rho)(1-\alpha)g'(1-\bar{\phi}) = 0$ by the first-order condition defining $\bar{\phi}$. It follows that

$$\Psi'(\alpha) = (\tfrac{1}{2})[g(0) - g(1)] - [\rho g(\bar{\phi}) - (1-\rho)g(1-\bar{\phi})]. \tag{B.8}$$

Next note that $\Psi(\alpha) = 0$ can be rearranged to yield

$$(\tfrac{1}{2})[g(0) - g(1)] - [\rho g(\bar{\phi}) - (1-\rho)g(1-\bar{\phi})]$$
$$= (1/\alpha)[(1-\rho)g(1-\bar{\phi}) - (\tfrac{1}{2})g(1)].$$

Substituting this into (B.8):

$$\Psi'(\alpha) = (1/\alpha)[(1-\rho)g(1-\bar{\phi}) - (\tfrac{1}{2})g(1)] > 0$$

as $g(1-\bar{\phi}) \geqslant g(1)$ and $1 - \rho > \frac{1}{2}$.

To summarize, $(x_1, x_2) = (0,0)$ is a Nash equilibrium iff $\Psi(\alpha) \geqslant 0$. It is been shown that $\Psi(\frac{1}{2}) < 0 < \Psi(1)$ and if $\Psi(\alpha) = 0$ then $\Psi'(\alpha) > 0$. There then exists a unique value of $\alpha$ over $(\frac{1}{2}, 1)$, denoted $\underline{\alpha}$, such that $\Psi(\alpha) \geqslant 0$ iff $\alpha \in [\underline{\alpha}, 1]$. □

**Proof of Theorem 3.** Consider $(x_1, x_2, \beta, \gamma)$ as a candidate absorbing state and suppose $x_1 < x_2$. Given this ordering, some consumers will switch loyalties unless all type 0 consumers are loyal to firm 1 and all type 1 consumers are loyal to firm 2. Thus, if $(x_1, x_2, \beta, \gamma)$ is an absorbing state and $x_1 < x_2$ then $(\beta, \gamma) = (\alpha, 0)$. Given $(\beta, \gamma) = (\alpha, 0)$, firms will not switch to any other location if $(x_1, x_2) = (\underline{\phi}, \bar{\phi})$ as then each firm's location is optimal given its customer base. Furthermore, if $x_1 \neq \underline{\phi}$ then firm 1 will change its location with positive probability; specifically, it will change to $\underline{\phi}$ if it receives such an idea. This is analogously true for firm 2 if $x_2 \neq \bar{\phi}$. We conclude that $(\underline{\phi}, \bar{\phi}, \alpha, 0)$ is the only absorbing state for which $x_1 < x_2$. By symmetry, $(\bar{\phi}, \underline{\phi}, 0, 1-\alpha)$ is the only absorbing state for which $x_1 > x_2$.

Next consider $(x_1, x_2, \beta, \gamma)$ as a candidate absorbing state and suppose $x_1 = x_2$. Given that firms are identical, half of all searching consumers will switch loyalties. Thus, $(\beta, \gamma)$ will change unless $(\beta, \gamma) = (\alpha/2, (1-\alpha)/2)$ so that half of all type 0 consumers and half of all type 1 consumers are loyal to firm 1. Thus, if $(x_1, x_2, \beta, \gamma)$ is an absorbing state and $x_1 = x_2$ then $(\beta, \gamma) = (\alpha/2, (1-\alpha)/2)$. Given $(\beta, \gamma) = (\alpha/2, (1-\alpha)/2)$, $(x_1, x_2)$ is stable iff $(x_1, x_2) = (\hat{\phi}, \hat{\phi})$ as $\hat{\phi}$ is the unique optimum given a customer base of $\alpha/2$ of type 0 consumers and $(1-\alpha)/2$ of type 1 consumers. We conclude that the unique absorbing state in which firms have identical locations is $(\hat{\phi}, \hat{\phi}, \alpha/2, (1-\alpha)/2)$. □

**Proof of Theorem 4.** Only the proof for when $(x_1^0, x_2^0, \beta^1, \gamma^1) \in \bar{\Omega}_1$ is provided as the proof for when the state lies in $\bar{\Omega}_2$ is similar. Recall that, with probability one, $x_i^t$ lies in

$$X \cap [\min\{\psi_i(x_i^{t-1}, \beta^t, \gamma^t), x_i^{t-1}\}, \max\{\psi_i(x_i^{t-1}, \beta^t, \gamma^t), x_i^{t-1}\}],$$

where $[\min\{\psi_i(x_i^{t-1}, \beta^t, \gamma^t), x_i^{t-1}\}, \max\{\psi_i(x_i^{t-1}, \beta^t, \gamma^t), x_i^{t-1}\}]$ is the set of ideas that yield virtual profit at least as great as $x_i^{t-1}$. Hence, if $\max\{\psi_1(x_1^{t-1}, \beta^t, \gamma^t), x_1^{t-1}\} < \min\{\psi_2(x_2^{t-1}, \beta^t, \gamma^t), x_2^{t-1}\}$ then $x_1^{t-1} < x_2^t$ and $x_1^t < x_2^{t-1}$ in which case those type 0 (1) consumers who are searching will end up being loyal to firm 1 (2). This is summarized as Lemma B.1; the proof of which is omitted since it is immediate.

**Lemma B.1.** *If* $\max\{\psi_1(x_1^{t-1}, \beta^t, \gamma^t), x_1^{t-1}\} < \min\{\psi_2(x_2^{t-1}, \beta^t, \gamma^t), x_2^{t-1}\}$ *then, with probability one,* $(\beta^{t+1}, \gamma^{t+1}) = (\beta^t + \rho(\alpha - \beta^t), (1-\rho)\gamma^t)$, $\phi_1(\beta^{t+1}, \gamma^{t+1}) < \phi_1(\beta^t, \gamma^t)$, *and* $\phi_2(\beta^t, \gamma^t) < \phi_2(\beta^{t+1}, \gamma^{t+1})$.

Lemma B.2 establishes that $\bar{\Omega}_1$ is a closed set of states.

**Lemma B.2.** *If*

$$\max\{\psi_1(x_1^{t-1}, \beta^t, \gamma^t), x_1^{t-1}\} < \min\{\psi_2(x_2^{t-1}, \beta^t, \gamma^t), x_2^{t-1}\} \tag{B.9}$$

*then*

$$\max\{\psi_1(x_1^{\tau}, \beta^{\tau+1}, \gamma^{\tau+1}), x_1^{\tau}\} < \min\{\psi_2(x_2^{\tau}, \beta^{\tau+1}, \gamma^{\tau+1}), x_2^{\tau}\}, \quad \forall \tau \geqslant t. \tag{B.10}$$

**Proof.** (B.9) implies $x_1^{t-1} < x_2^{t-1}$. The proof strategy is to show that if (B.9) holds then

$$\max\{\psi_1(x_1^t, \beta^{t+1}, \gamma^{t+1}), x_1^t\} \leqslant \max\{\psi_1(x_1^{t-1}, \beta^t, \gamma^t), x_1^{t-1}\}. \tag{B.11}$$

We will then claim that, by an analogous argument, one can show that if (B.9) holds then

$$\min\{\psi_2(x_2^{t-1}, \beta^t, \gamma^t), x_2^{t-1}\} \leqslant \min\{\psi_2(x_2^t, \beta^{t+1}, \gamma^{t+1}), x_2^t\}. \tag{B.12}$$

Lemma B.2 follows by induction. To save on notation, let $\psi_i^t$ denote $\psi_i(x_1^{t-1}, \beta^t, \gamma^t)$ and $\phi_i^t$ denote $\phi_i(\beta^t, \gamma^t)$. Recall that $\phi_i^t$ is the value for $x_i^t$ from [0,1] that maximizes firm $i$'s virtual profit and that it lies in the interior $[\min\{\psi_i^t, x_i^{t-1}\}, \max\{\psi_i^t, x_i^{t-1}\}]$ with the exception when $\min\{\psi_i^t, x_i^{t-1}\} = \max\{\psi_i^t, x_i^{t-1}\}$ or, equivalently, $\psi_i^t = \phi_i^t = x_i^{t-1}$. In that exceptional case, it follows that $x_1^t = x_1^{t-1}$ and thus (B.11) holds. The remainder of the proof will deal with when $\psi_i^t \neq x_i^{t-1}$.

Note that since $x_1^t \leqslant \max\{\psi_1^t, x_1^{t-1}\}$ then a sufficient condition for (B.11) to be true is $\psi_1^{t+1} \leqslant \max\{\psi_1^t, x_1^{t-1}\}$. It is also useful to note that it follows from Lemma B.1 that $\phi_1^t < \phi_1^{t+1}$.

(i) Suppose $\psi_1^t < x_1^{t-1}$. This case is partitioned into two sub-cases.

(i(a)) Suppose $\phi_1^{t+1} \leqslant \psi_1^t$. This condition plus the fact that $\psi_1^t \leqslant x_1^t$ – which follows from $x_1^t \in [\psi_1^t, x_1^{t-1}]$ – implies $\phi_1^{t+1} \leqslant x_1^t$. It then follows that $\psi_1^{t+1} \leqslant \phi_1^{t+1}$. Therefore, $\psi_1^{t+1} \leqslant x_1^t \leqslant \max\{\psi_1^t, x_1^{t-1}\}$ and thus $\max\{\psi_1^{t+1}, x_1^t\} \leqslant \max\{\psi_1^t, x_1^{t-1}\}$.

(i(b)) Suppose $\psi_1^t < \phi_1^{t+1}$. To handle this case, two properties are required. First, if $x_1 < \phi_1(\beta,\gamma)$ then $\psi_1(x_1,\beta,\gamma)$ is non-increasing in $x_1$. Recall that $\psi_1(x',\beta,\gamma)$ is implicitly defined by

$$\tilde{\pi}_1(x',\beta,\gamma) = \tilde{\pi}_1(\psi_1(x',\beta,\gamma),\beta,\gamma), \tag{B.13}$$

when such a solution exists in [0,1] and otherwise is a corner solution. In the latter case, $\psi_1(x_1',\beta,\gamma)$ is fixed with respect to $x_1$. When $\psi_1(x',\beta,\gamma) \in (0,1)$, totally differentiating (B.13) and solving for $\partial\psi_1(x',\beta,\gamma)/\partial x_1$ yields

$$\frac{\partial\psi_1(x',\beta,\gamma)}{\partial x_1} = \frac{\partial\tilde{\pi}_1(x',\beta,\gamma)/\partial x_1}{\partial\tilde{\pi}_1(\psi_1(x',\beta,\gamma),\beta,\gamma)/\partial x_1},$$

which is negative by (B.13) and that $\tilde{\pi}_1(x,\beta,\gamma)$ is concave with respect to $x_1$. The second needed property is if $\beta'' \geqslant \beta'$ and $\gamma'' \leqslant \gamma'$ then $\psi_1(x_1,\beta'',\gamma'') \leqslant \psi_1(x_1,\beta',\gamma')$. Recall that virtual profit takes the form

$$\tilde{\pi}_1(x_1,\beta,\gamma) = \theta_0(\beta,\gamma)g(x) + \theta_1(\beta,\gamma)g(1-x),$$

where we have replaced $(w(0), w(1))$ with $(\beta,\gamma)$. Using this expression in (B.13) and totally differentiating with respect to $\beta$ yields, after solving for $\partial\psi_1(x',\beta',\gamma')/\partial\beta$,

$$\partial\psi_1(x',\beta',\gamma')/\partial\beta = \frac{(\partial\theta_0/\partial\beta)[g(x') - g(\psi_1(x',\beta',\gamma'))] - (\partial\theta_1/\partial\beta)[g(1-x') - g(1-\psi_1(x',\beta',\gamma'))]}{\theta_0(\beta,\gamma)g'(\psi_1(x',\beta',\gamma')) - \theta_1(\beta,\gamma)g'(1-\psi_1(x',\beta',\gamma'))} \tag{B.14}$$

If $x' < \psi_1(x',\beta',\gamma')$ then the denominator is negative as it is $\partial\tilde{\pi}_1(\psi_1(x',\beta,\gamma),\beta,\gamma)/\partial x_1$.

Since $x' < \psi_1(x',\beta',\gamma')$ also implies $g(x') > g(\psi_1(x',\beta',\gamma'))$ and $g(1-x') < g(1-\psi_1(x',\beta',\gamma'))$, it then follows from $\partial\theta_0/\partial\beta > 0$ and $\partial\theta_1/\partial\beta \leqslant 0$ (by A.5) that the numerator is positive. Hence, (B.14) is negative when $x' < \psi_1(x',\beta',\gamma')$. An analogous proof applies when $x' > \psi_1(x',\beta',\gamma')$. We conclude that $\psi_1(x_1,\beta,\gamma)$ is non-increasing in $\beta$.[22] An analogous proof shows that $\psi_1(x_1,\beta,\gamma)$ is non-decreasing in $\gamma$. Therefore, if $\beta'' \geqslant \beta'$ and $\gamma'' \leqslant \gamma'$ then $\psi_1(x_1,\beta'',\gamma'') \leqslant \psi_1(x_1,\beta',\gamma')$.

With these two properties, we can show that if $\psi_1^t < x_1^{t-1}$ and $\psi_1^t < \phi_1^{t+1}$ then $\psi_1^{t+1} \leqslant \max\{\psi_1^t, x_1^{t-1}\}$. Since $x_1^t \in [\psi_1^t, x_1^{t-1}]$ and $\psi_1^{t+1}$ is non-increasing in $x_1^t$ then $\psi_1^{t+1}$ is maximized when $x_1^t = \psi_1^t$: $\psi_1^{t+1} \leqslant \psi_1(\psi_1^t, \beta^{t+1}, \gamma^{t+1})$. Given $\psi_1$ is non-increasing in $\beta$ and non-decreasing in $\gamma$, it then follows from $\beta^{t+1} \geqslant \beta^t$ and $\gamma^{t+1} \leqslant \gamma^t$ that $\psi_1(\psi_1^t, \beta^{t+1}, \gamma^{t+1}) \leqslant \psi_1(\psi_1^t, \beta^t, \gamma^t)$ and thus $\psi_1^{t+1} \leqslant \psi_1(\psi_1^t, \beta^t, \gamma^t)$. Finally, since $x_1^{t-1} = \psi_1(\psi_1^t, \beta^t, \gamma^t)$, we conclude that $\psi_1^{t+1} \leqslant x_1^{t-1}$. Given that it was assumed $\psi_1^t < x_1^{t-1}$, $\psi_1^{t+1} \leqslant \max\{\psi_1^t, x_1^{t-1}\}$.

(ii) Suppose $x_1^{t-1} < \psi_1^t$. This case is partitioned into two sub-cases.

(ii(a)) Suppose $\phi_1^{t+1} \leqslant x_1^{t-1}$. Since $x_1^t \in [x_1^{t-1}, \psi_1^t]$, it follows that $\phi_1^{t+1} \leqslant x_1^t$ and, therefore, $\psi_1^{t+1} \leqslant x_1^t$. Hence, $\psi_1^{t+1} \leqslant \max\{\psi_1^t, x_1^{t-1}\}$.

(ii(b)) Suppose $x_1^{t-1} < \phi_1^{t+1}$. The proof is analogous to that in (i(b)). The maximal value for $\psi_1^{t+1}$ is $\psi_1(x_1^{t-1}, \beta^{t+1}, \gamma^{t+1})$ and since $\psi_1(x_1^{t-1}, \beta^{t+1}, \gamma^{t+1}) \leqslant \psi_1(x_1^{t-1}, \beta^t, \gamma^t)$ then $\psi_1^{t+1} \leqslant \psi_1(x_1^{t-1}, \beta^t, \gamma^t)$. This proves $\psi_1^{t+1} \leqslant \max\{\psi_1^t, x_1^{t-1}\}$. □

Using Lemmas B.1–B.2, we can prove Theorem 4 by induction. As $\bar{\Omega}_1$ is a closed set of states, it follows from Lemma B.1 that if $\max\{\psi_1^t, x_1^{t-1}\} < \min\{\psi_2^t, x_2^{t-1}\}$ then

$$(\beta^{\tau+1}, \gamma^{\tau+1}) = (\beta^\tau + \rho(\alpha - \beta^\tau), (1-\rho)\gamma^\tau) \ \forall \tau \geqslant t.$$

We claim that this implies $\beta^\tau = (1-\rho)^{\tau-t}\beta^t + [1-(1-\rho)^{\tau-t}]\alpha$ and $\gamma^\tau = (1-\rho)^{\tau-t}\gamma^t$. Suppose it is true for $\tau'$. It follows from Lemma B.1 that

$$\begin{aligned}
\beta^{\tau'+1} &= \beta^{\tau'} + \rho(\alpha - \beta^{\tau'}) \\
&= \{(1-\rho)^{\tau'-t}\beta^t + [1-(1-\rho)^{\tau'-t}]\alpha\} \\
&\quad + \rho\{\alpha - (1-\rho)^{\tau'-t}\beta^t - [1-(1-\rho)^{\tau'-t}]\alpha\} \\
&= (1-\rho)^{\tau'+1-t}\beta^t + [1-(1-\rho)^{\tau'+1-t}]\alpha, \qquad \text{(B.15)}
\end{aligned}$$

which establishes the dynamic path for $\beta^\tau$. Now suppose $\gamma^\tau = (1-\rho)^{\tau-t}\gamma^t$:

$$\gamma^{\tau'+1} = (1-\rho)\gamma^{\tau'} = (1-\rho)[(1-\rho)^{\tau-t}\gamma^t] = (1-\rho)\gamma^{\tau'+1}, \qquad \text{(B.16)}$$

which establishes the dynamic path for $\gamma^\tau$. Thus, if $(x_1^0, x_2^0, \beta^1, \gamma^1) \in \bar{\Omega}_1$ then $\lim_{t\to\infty} (\beta^t, \gamma^t) = (\alpha, 0)$.

Once $(\beta^t, \gamma^t)$ is sufficiently close to $(\alpha, 0)$, the unique optimum from $X$ for firm 1 is $\underline{\phi}$ and for firm 2 is $\bar{\phi}$. To see why this is true, consider firm 1 (with the same logic applying to firm 2). From Lemma 1, $\underline{\phi}$ is the unique optimum from [0,1] when $(\beta,\gamma) = (\alpha, 0)$. Given that $X$ is finite and $\underline{\phi} \in X$, then $\underline{\phi}$ is the unique optimum in a neighborhood of $(\alpha, 0)$. Given $(\beta^t, \gamma^t)$ is converging to $(\alpha, 0)$, $\exists$ finite $T$ such that if $t > T$ then if firm 1 generates idea $\underline{\phi}$ it will adopt it and, in addition, will maintain that location in all

[22] The preceding analysis presumed $\psi_1(x_1,\beta,\gamma) \in (0,1)$. If it is instead a corner solution then it is locally independent of $\beta$.

ensuring periods since the distance between $(\beta^t, \gamma^t)$ and $(\alpha, 0)$ is shrinking. Similarly, if firm 2 generates idea $\bar{\phi}$, it will adopt it and maintain that location in all ensuring periods. Note that $T$ is bounded and this bound can be set independent of $(\beta^0, \gamma^0)$. Since there is a positive probability in each period of firm 1 generating idea $\underline{\phi}$ and firm 2 generating idea $\bar{\phi}$ then, with probability one, $\lim_{t\to\infty}(x_1^t, x_2^t) = (\bar{\phi}, \underline{\phi})$. This completes the proof. □

**Proof of Theorem 5.** Define $T$ as the minimum number of periods such that, when firm 1's location remains less than firm 2's location, firm 1's base of loyal type 0 customers is sufficiently close to $\alpha$ and of loyal type 1 customers is sufficiently close to 0 that $\underline{\phi}$ is firm 1's optimal location in $X$ and $\bar{\phi}$ is firm 2's optimal location in $X$. Note that $T$ can be defined independent of the initial customer allocation. Now suppose $x_1^0 < x_2^0$ and consider the following event which occurs with positive probability: no new ideas over the next $T$ periods and, in the ensuring period, firm 1 generates idea $\underline{\phi}$ and firm 2 generates idea $\bar{\phi}$.[23] Firms will adopt those ideas. At that point, the state is in $\bar{\Omega}_1$. By Theorem 4, the system converges to $(\underline{\phi}, \bar{\phi}, \alpha, 0)$. Analogously, one can show that if $x_1^0 > x_2^0$ then the system converges to $(\bar{\phi}, \underline{\phi}, 0, 1-\alpha)$ with positive probability. □

**Proof of Theorem 6.** When firms qualities are identical, which is an event that can occur for any finite length of time (by A7), firms' locations and customer bases evolve exactly as found in Section 4. Hence, the only candidates for absorbing pairs of locations are those described in Theorem 3. For the location pair $(\underline{\phi}, \bar{\phi})$, consider $\Lambda = \{(\alpha, 0)\}$. Since $|z^t| < \bar{\phi} - \underline{\phi}$ then type 0 consumers prefer firm 1 to firm 2 and type 1 consumers prefer firm 2 to firm 1 regardless of the quality differential. Hence, if $(\beta, \gamma) = (\alpha, 0)$ then there is no change in customer allocations. Furthermore, given those customer allocations, firms have no desire to change their locations from $(\underline{\phi}, \bar{\phi})$. $(\underline{\phi}, \bar{\phi})$ is then an absorbing pair of locations. By symmetry, $(\bar{\phi}, \underline{\phi})$ is an absorbing pair of locations.

Now consider $(\hat{\phi}, \hat{\phi})$ and suppose $\Lambda = \{(\beta, \gamma)\colon \beta/\gamma = \alpha/(1-\alpha)\}$. If $\beta/\gamma = \alpha/(1-\alpha)$ then, by the definition of $\hat{\phi}$ and A8, each firm's location is optimal. Of course, customer allocations can change due to quality differences. What we just need for $(\hat{\phi}, \hat{\phi})$ to be an absorbing pair of locations is if $(\beta^t, \gamma^t) \in \Lambda$ then $(\beta^{t+1}, \gamma^{t+1}) \in \Lambda$. Suppose $z^t > 0$ so that all consumer types prefer firm 1. The equations of motion on customer allocations are

$$\beta^{t+1} = \beta^t + \rho(\alpha - \beta^t),$$

$$\gamma^{t+1} = \gamma^t + \rho(1 - \alpha - \gamma^t).$$

Using these expressions, it is straightforward to show that if $\beta^t/\gamma^t = \alpha/(1-\alpha)$ then $\beta^{t+1}/\gamma^{t+1} = \alpha/(1-\alpha)$. This can be similarly done for when $z^t = 0$ and $z^t < 0$ by using the appropriate equations of motion. Therefore, if $(x_1^{t-1}, x_2^{t-1}) = (\hat{\phi}, \hat{\phi})$ and $(\beta^t, \gamma^t) \in \{(\beta, \gamma)\colon$

[23] If one assumed that an idea was generated each period for sure then the required sequence would be of ideas that are not superior to a firm's current location which is also an event with positive probability.

$\beta/\gamma = \alpha/(1-\alpha)\}$ then, regardless of the quality differential, $(x_1^t, x_2^t) = (\hat{\phi}, \hat{\phi})$ and $(\beta^{t+1}, \gamma^{t+1}) \in \{(\beta, \gamma)\colon \beta/\gamma = \alpha/(1-\alpha)\}$ for sure. $\square$

**Proof of Theorem 7.** It is sufficient to suppose $\beta^{t'} + \gamma^{t'} \geqslant \frac{1}{2}$ since, by symmetry, the same line of argument works when $\beta^{t'} + \gamma^{t'} < \frac{1}{2}$. First consider the case of $x_1^{t'-1} < x_2^{t'-1}$. $\exists$ finite $T$ such that, with positive probability, $\beta^{t'+T}$ can be made sufficiently close to $\alpha$ so that $\beta^{t'+T} > \frac{1}{2}$ and thus $\beta^{t'+T} + \gamma^{t'+T} > \frac{1}{2}$. The event that allows this to happen is $z^t \geqslant 0\, \forall t \in \{t', \dots, t'+T\}$ and firms receive no new ideas over those periods. To show that $\exists$ finite $T$ such that, with positive probability, $\beta^{t'+T} + \gamma^{t'+T} < \frac{1}{2}$, first consider a sequence of $T'$ periods whereby the quality differential is zero and firms' locations do not change. Then suppose firm 1 generates idea $\underline{\phi}$ and firm 2 generates idea $\bar{\phi}$. $T'$ can be chosen so that firms adopt those locations. Now suppose that the quality differential takes a value less than $-(\bar{\phi} - \underline{\phi})$ so that both consumer types are flowing to firm 2. Letting that situation persist sufficiently long, we can make firm 2's market share exceed $\frac{1}{2}$.

When $x_1^{t'-1} > x_2^{t'-1}$, a sequence of length $T$ of no new ideas and a zero quality differential will result in $\beta^{t'+T} + \gamma^{t'+T} < \frac{1}{2}$ when $T$ is sufficiently high. To get $\beta^{t'} + \gamma^{t'} \geqslant \frac{1}{2}$, assume a sequence with a zero quality differential so that type 0 consumers are flowing to firm 2 and type 1 consumers to firm 1. Once the customer allocation is sufficiently close to $(0, 1-\alpha)$, let firms have ideas so that their new locations are $(\bar{\phi}, \underline{\phi})$. At that point, assume a quality shock that exceeds $\bar{\phi} - \underline{\phi}$ so that both consumer types are flowing to firm 1. By assuming this quality differential is maintained without any change in locations, eventually firm 1's share of type 0 consumers will exceed $\frac{1}{2}$.

Finally, consider the case of $x_1^{t'-1} = x_2^{t'-1}$. With positive probability, firms will receive no new ideas over $T$ periods and firm 1 will have superior quality. Given $\beta^{t'} + \gamma^{t'} \geqslant \frac{1}{2}$ then $\beta^{t'+T} + \gamma^{t'+T} \geqslant \frac{1}{2}$. Suppose instead that firm 2 experiences superior quality for $T$ periods. Given that both consumer types are flowing to firm 2, one can choose $T$ long enough so that $\beta^{t'+T} + \gamma^{t'+T} < \frac{1}{2}$. $\square$

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