The Journal of Industrial Economics

WILEY

ORIGINAL ARTICLE

Hub-and-Spoke Collusion With a Third-Party Pricing Algorithm

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Accepted: 9 December 2025

Keywords: antitrust | collusion | pricing algorithms

ABSTRACT

A data analytics company delivers an efficiency by supplying a pricing algorithm that allows prices to more effectively respond to demand variation. In this setting, I consider a new form of hub-and-spoke collusion: A data analytics company (hub) coordinates the prices of competitors (spokes) through its pricing algorithm. A novel finding is that the data analytics company's efficiency is a facilitating factor for collusion; a rise in this efficiency increases the supracompetitive markup and the incremental profit from collusion. Thus, markets in which a third party's services are solidly grounded in efficiency still warrant scrutiny by competition authorities because they are more prone to the emergence of collusion and anticompetitive harm.

1 | Introduction

RealPage and Yardi are data analytics companies that supply a pricing algorithm to assist apartment owners in setting rents. IDeaS and Rainmaker are data analytics companies that supply a pricing algorithm to assist hotel chains in setting room rates. In a series of recent legal cases in the United States, plaintiffs claim that each of these data analytics companies has an unlawful price-fixing agreement with those firms using its pricing algorithm. Quoting from two of the complaints:

> RealPage provides software and data analytics to Lessors and serves as the mechanism by which Lessors collude and avoid competition, increasing lease prices to Plaintiffs … RealPage has been facilitating an agreement among participating Lessors not to compete on price …[1]

> … Operator Defendants have agreed and conspired to outsource their independent pricing decision-making to a single, common pricing manager, IDeaS, which has willingly facilitated and enforced the conspiracy.[2]

Claimed is the presence of a hub-and-spoke cartel where the data analytics company is the hub coordinating the prices of subscribing firms, who are the spokes. The possibility of hub-and-spoke collusion involving such a third parties was noted in Ezrachi and Stucke [1] with concerns expressed by various institutions including the UK's Competition & Markets Authority,[3] the German Monopolies Commission,[4] and the OECD.[5] In the United States, these concerns have led the U.S. Senate, as well as several cities and states, to propose or pass laws that would restrict data sharing between a data analytics company and subscribing firms. The expressed purpose of these laws is to prevent anticompetitive agreements.[6]

The objective of this study is to theoretically explore collusion in this setting. Though there is an extensive body of research on collusion, with some of it considering hub-and-spoke cartels, it does not capture the essential features when a data analytics company is involved. As described in Garrod et al. [5], a hub is most often an upstream supplier, such as a manufacturer or wholesaler supplying retailers. Similarly, a data analytics company is an upstream supplier, but its input is a pricing algorithm rather than a consumer product or an input to be used in a production process. While an upstream firm's supply of an input is typically orthogonal to its role in coordinating customers' prices, the supply of a pricing algorithm is intrinsically connected to the implementation of price coordination. Consequently, a new theory is needed to understand hub-and-spoke collusion when the hub is a data analytics company.

This paper addresses several questions: What is the structure of a stable and profitable collusive scheme between a data analytics company and adopting firms? What market conditions are conducive to collusion? What are the determinants of the supracompetitive markup? To explore these questions, a model is developed with many small firms, which could represent some geographic markets for apartment rentals. There is a source of demand variation that could be across time or represent a collection of submarkets. Firms are assumed to lack the information or expertise to condition price on this demand variation. The efficiency delivered by the third party is that it can supply a pricing algorithm that conditions the price on the demand state. As shown in Harrington [6, 7], when there is no collusive agreement, the third party will choose a pricing algorithm that, in comparison to what firms would design on their own (if they could), has a price that is more sensitive to demand but with the same average price. Even though adopting firms are using the same third-party pricing algorithm and which is designed to maximize the third party's profit from selling it, the pricing algorithm does not embody a supracompetitive markup. That setup is modified in this paper to allow the third party and adopting firms to have a collusive agreement. Specifically, the third party designs the pricing algorithm to maximize adopting firms' profits and, in exchange, the firms pay a higher fee to the third party. As there are many firms, a collusive scheme that involves threatening cartel collapse or a temporary price war in response to a firm's noncompliance would seem impractically fragile because of its sensitivity to a single firm's conduct. Consequently, the scheme that is explored is one in which each firm decides whether to participate in the arrangement. As a result, the third party's problem under collusion is to maximize adopting firms' profits subject to a participation constraint that makes it incentive compatible for firms to adopt the pricing algorithm. Upon adopting the pricing algorithm, a firm is assumed to implement the prices generated by the pricing algorithm.

The theory of collusion delivers some novel findings, which are due to the third party not just coordinating firms' prices but also delivering an efficiency in the form of conditioning price on the demand state. The first finding is that collusion is effective even when there are many small (actually, negligible) firms. When participation in a collusive scheme is endogenous, as is considered here, current understanding is that small firms would prefer to be outside the cartel, undercutting the collusive price rather than inside the cartel, constraining their conduct.[7] In contrast, collusion with a data analytics company acting as a hub does have small firms participating. While a firm has the option not to participate and instead charge a uniform price, which undercuts the average collusive price of adopting firms, it would forego the efficiency coming from conditioning the price on the demand state. This efficiency can induce participation. Hence, supracompetitive prices can be stable even for a market structure that is generally inhospitable to collusion.

At the same time, the efficiency poses a constraint on the supracompetitive markup generated by the third party's pricing algorithm. Setting the markup too high could induce firms not to adopt, as they find it more profitable to charge a uniform price that undercuts adopting firms' average collusive price. We show that the supracompetitive markup is increasing in the efficiency delivered by the third party. The greater the efficiency, the more willing firms are to adopt the pricing algorithm, and that can be leveraged by the third party to set a higher markup. Given that the incremental profit from collusion is then higher, collusion is then predicted to be more likely when the efficiency is larger. To my knowledge, this is the first theory to show an efficiency to be a facilitating factor in collusion, which, as will be discussed later, poses some challenges for the effective enforcement of competition law.

1.1 | Related Literature

There is a growing literature exploring the implications of a firm adopting and committing to a pricing algorithm. Though not encompassing a third-party supplier of pricing algorithms, work includes Arunachaleswaran et al. [9], Brown and MacKay [10, 11], Lamba and Zhuk [12], Leisten [13], and Salcedo [14]. Musolff [15] has third parties in the form of repricers who supply pricing algorithm templates to firms. As it is the decision of a firm whether to make its pricing algorithm, say, aggressive or accommodating, the repricers' offerings do not determine the degree of competition as with the settings motivating my analysis. Johnson et al. [16] consider a third party in the form of a platform that restricts the space of feasible pricing algorithms. In their setting, the platform is acting to *prevent* collusion as opposed to the current setting where a third party is acting to *facilitate* collusion. A platform's incentives are quite distinct from a third party whose profits come from selling its pricing algorithm.

When a third party is supplying a pricing algorithm that responds to demand variation, firms have been modeled as deciding whether to adopt the pricing algorithm or, after having adopted the pricing algorithm, whether to implement the algorithm's generated prices.[8] Investigating information disclosure by a third party, Hickok [18] and Sugaya and Wolitzky [19] endogenize the price implementation decision while assuming firms adopt the pricing algorithm. Previous work by the author endogenizes the adoption issue while assuming firms implement recommended prices. Harrington [6, 7] has the third party and firms act independently, so there is no collusive agreement. Harrington [20] compares the third party's pricing algorithm when it has a collusive agreement with subscribing firms and when it does not. The current paper modifies the collusive model in Harrington [20] by endogenizing firms' adoption decisions. This introduces a new force as reflected in a novel theory of how a third party's efficiency influences the anticompetitive harm from collusion. Finally, there are two empirical studies of the effect of a third party supplying pricing algorithms. Evidence of anticompetitive effect is found for the German retail gasoline market in Assad et al. [21], while both procompetitive and anticompetitive effects are found for the U.S. apartment rentals market in Calder-Wang and Kim [22].

There is some related work on how collusion is impacted by firms' knowledge of demand, though the models are not suitable for encompassing a data analytics company. One line considers hub-and-spoke collusion with the third party supplying downstream retailers who have private information on demand. In Sahuguet and Walckiers [23], the spokes know demand and may decide to share their knowledge with the hub. In Do [24], the downstream firms receive demand signals which they can share with the upstream firm. These papers explore the potential advantage of downstream firms colluding with the assistance of a hub. While not involving a third party, there is also a literature exploring how collusion is affected when firms are better informed about the demand state, as could occur when using algorithmic pricing; see Miklós-Thal and Tucker [25], O'Connor and Wilson [26], and Martin and Rasch [27].

1.2 | Roadmap

In Section 2, competition among firms is characterized without and with a third party supplying a pricing algorithm. A model of collusion between the third party and firms using its pricing algorithm is presented in Section 3 with the collusive equilibrium characterized in Section 4. Comparative statics are performed in Section 5 to explore the determinants of the collusive markup. Section 6 discusses some implications for competition law and policy, and Section 7 concludes. All proofs are in the Appendix A.

2 | Competitive Model and Equilibrium[9]

2.1 | Without a Third-Party Developer

Consider a market where a continuum of firms have symmetrically differentiated products and a common and constant marginal cost c. A firm's demand function is linear, $a - bp_i + dP_{-i}$, where p_i is own price, P_{-i} is the average price of rival firms, and $b > d > 0$.[10] A critical feature is demand variation with respect to a which has a continuously differentiable cdf $G : [\underline{a}, \overline{a}] \to [0,1]$ with mean μ and variance σ^2. The higher is a, the stronger is firm demand in the sense that the amount of demand is larger and demand is more price-inelastic. Assume $\underline{a} - (b-d)c > 0$ and $|\overline{a} - \underline{a}|$ is not too large, so, at equilibrium prices, demand is positive for all demand states. The demand state a has two interpretations. A firm could face a single demand curve and a is subject to a demand shock which results in the distribution G. In that case, the price may condition on the current demand shock. Alternatively, a firm faces a collection of market segments represented by G, in which case price may condition on the market segment a.

Assume the demand state a is realized at a higher frequency than a firm's pricing decisions or, when G represents a collection of market segments, the firm cannot distinguish among those segments when pricing. Thus, a firm cannot condition the price on the demand state. A symmetric Nash equilibrium price p^N is defined by:

$$p^N = \arg\max_{p_i \in \Re_+} \int (p_i - c)\left(a - bp_i + dp^N\right) G'(a)da$$
$$\Rightarrow p^N = \frac{\mu + bc}{2b - d}.$$

As explored in Section 2.2, the deliverable of a third party is that it can provide a pricing algorithm that is able to track the demand state. In assessing the impact of firms outsourcing their pricing algorithms, a useful benchmark is when firms are able to design their own pricing algorithms to condition price on the demand state. When a fraction θ of firms can do so, it is straightforward to show their Nash equilibrium pricing algorithm is

$$\frac{d(1-\theta)\mu + (2b - d\theta)bc}{(2b - d\theta)(2b - d)} + \left(\frac{1}{2b - d\theta}\right)a,$$

while a fraction $1 - \theta$ of firms set a uniform price equal to $\frac{\mu+bc}{2b-d}$. Note that

$$E_a\left[\frac{d(1-\theta)\mu + (2b - d\theta)bc}{(2b - d\theta)(2b - d)} + \left(\frac{1}{2b - d\theta}\right)a\right] = \frac{\mu + bc}{2b - d}$$

so the average price is the same for all firms.

2.2 | With a Third-Party Developer

Let us now introduce a third-party developer whose business model is to design and sell pricing algorithms. The efficiency delivered by the third party is that its pricing algorithm can track the high-frequency demand shock (or submarkets) and thus condition price on the demand state a. The third party chooses a pricing algorithm $\phi : [\underline{a}, \overline{a}] \to \Re_+$ to maximize its profit given firms' expected adoption decisions. I will assume the fee f paid by an adopting firm to the third party is some fraction $\eta \in (0, 1]$ of a firm's willingness-to-pay (WTP).[11] A fraction θ of firms are endowed with the option to pay f and adopt the pricing algorithm ϕ. As the fee is proportional to the WTP, all θ firms will end up adopting, in equilibrium. A fraction $1 - \theta$ of firms are assumed not to have this option, perhaps because they are not open to the idea of delegating pricing to a third party.[12] By not requiring $\theta = 1$, comparative statics with respect to the adoption rate can be performed.[13] After adoption decisions are made, the demand state is realized, and firms choose prices. If a firm adopted the third party's pricing algorithm, then it prices at $\phi(a)$. For analytical tractability, firms are not allowed to override the algorithm's output, which is a plausible assumption for markets where the algorithm is adjusting prices at a high frequency—such as gasoline—so a firm may need to automate implementation.[14] If a firm did not adopt the third party's pricing algorithm, then it chooses a uniform price (i.e., price does not condition on the demand state), which maximizes its expected profits.

An equilibrium is defined by a pricing algorithm, adoption decisions, and a (symmetric) price for non-adopting firms, such that: (i) the third party's pricing algorithm is optimal given the firms' adoption decisions and the fee; (ii) each firm's adoption decision is optimal given rival firms' prices and the third party's pricing algorithm and fee; and (iii) each non-adopting firm's price is optimal given rival firms' prices. The formal conditions defining an equilibrium are provided below.

The third party's problem is to design the pricing algorithm to maximize its profit (or, equivalently, revenue, as there is assumed to be no marginal cost to supplying a pricing algorithm). The demand for its pricing algorithm is θ as long as the payoff from a firm adopting is at least as great as that from not adopting. Thus, the third party's problem is

$$\max_{\phi} \theta f(\phi) \text{ s.t. } \pi^{A}(\phi) - f(\phi) \geq \pi^{NA}(\phi), \tag{1}$$

where $\pi^{A}(\phi)$ is an adopter's profit, $\pi^{NA}(\phi)$ is a non-adopter's profit, $\pi^{A}(\phi) - \pi^{NA}(\phi)$ is the WTP, and $f(\phi) \equiv \eta\left(\pi^{A}(\phi) - \pi^{NA}(\phi)\right)$ is the fee. Simplifying (1), it is equivalent to:

$$\max_{\phi} \pi^{A}(\phi) - \pi^{NA}(\phi) \text{ s.t. } \pi^{A}(\phi) \geq \pi^{NA}(\phi). \tag{2}$$

Thus, the third party optimally designs the pricing algorithm to maximize the WTP (and the constraint will not bind).

With θ firms adopting, an equilibrium is a pricing algorithm and a uniform price for non-adopting firms, $\left(\phi^{I}, p^{I}\right)$, where ϕ^{I} maximizes the WTP for the pricing algorithm and p^{I} is a price that maximizes a non-adopter's profit.[15] Letting

$$\pi\left(p_i, P_{-i}, a\right) \equiv (p_i - c)\left(a - bp_i + dP_{-i}\right),$$

denote a firm's profit function, $\left(\phi^{I}, p^{I}\right)$ is a solution to:

$$\phi^{I} = \arg\max_{\phi} \int \pi\left(\phi(a), \theta\phi(a) + (1-\theta)p^{I}, a\right)G'(a)da - \int \pi\left(p^{I}, \theta\phi(a) + (1-\theta)p^{I}, a\right)G'(a)da \tag{3}$$

$$p^{I} = \arg\max_{p} \int \pi\left(p, \theta\phi^{I}(a) + (1-\theta)p^{I}, a\right)G'(a)da. \tag{4}$$

Equation (3) ensures ϕ^{I} maximizes the WTP given θ firms adopt and $1-\theta$ non-adopting firms price at p^{I}, and (4) ensures p^{I} maximizes a non-adopting firm's profit given $1-\theta$ rival firms price at p^{I} and θ rival firms price according to ϕ^{I}.[16]

In light of the linearity of demand, the solution to (3) and (4) is an affine pricing algorithm: $\phi(a) = \alpha + \gamma a$ for some (α, γ).[17] (3) and (4) now take the form:

$$\begin{aligned}\left(\alpha^{I}, \gamma^{I}\right) &= \arg\max_{(\alpha,\gamma)} \int (\alpha + \gamma a - c)(a - b(\alpha + \gamma a) + d(\theta(\alpha + \gamma a) \\ &\quad + (1-\theta)p^{I}))G'(a)da \\ &\quad - \int (p^{I} - c)\left(a - bp^{I} + d\left(\theta(\alpha + \gamma a) + (1-\theta)p^{I}\right)\right)G'(a)da,\end{aligned} \tag{5}$$

$$p^{I} = \arg\max_{p} \int (p-c)\left(a - bp + d\left(\theta\left(\alpha^{I} + \gamma^{I}a\right) + (1-\theta)p^{I}\right)\right)G'(a)da. \tag{6}$$

It is shown in Harrington [20] that the unique solution to (5) and (6) is:

$$\alpha^{I} = \frac{2bc(b-d\theta) + d\mu(1-2\theta)}{2(b-d\theta)(2b-d)},\ \gamma^{I} = \frac{1}{2(b-d\theta)},\ p^{I} = \frac{\mu + bc}{2b-d}. \tag{7}$$

Note

$$E_a\left[\frac{2bc(b-d\theta) + d\mu(1-2\theta)}{2(b-d\theta)(2b-d)} + \left(\frac{1}{2(b-d\theta)}\right)a\right] = \frac{\mu + bc}{2b-d}.$$

Hence, the average price is $\frac{\mu+bc}{2b-d}$ for both adopting firms and non-adopting firms, which is the same as the average equilibrium price without a third-party developer.

Let me explain why the third party does not build in a supracompetitive markup. It is true that a higher average price for the pricing algorithm would raise the profits for an adopting firm. That follows from all θ adopting firms' pricing closer to the joint profit-maximizing price. However, it would also raise the profits to a non-adopting firm because θ firms are pricing higher. Given a non-adopting firm optimizes its uniform price against the average price of adopting firms, while an adopting firm does not (as it is just setting the high average price imposed by the pricing algorithm), a non-adopting firm's profit rises *more* than an adopting firm's profit in response to an increase in the average price of the pricing algorithm. Consequently, a firm's WTP is reduced, so the third party has to charge a lower fee and thus earn lower revenue. While a third party could design the pricing algorithm to have a supracompetitive markup, it is optimal for it not to do so.[18]

Substituting the expressions for $\left(\alpha^{I}, \gamma^{I}\right)$ into the WTP in Equation (5), it is shown in Harrington [20] that the WTP equals $\frac{\sigma^2}{4(b-d\theta)}$ in which case the equilibrium fee is $f = \frac{\eta\sigma^2}{4(b-d\theta)}$.

3 | Collusive Model and Equilibrium Conditions

Suppose the initial state of the market is one of competition, as described in Section 2.2. The third party designed the pricing algorithm to maximize its profit from selling it, and θ firms adopted it while paying the equilibrium fee of $\frac{\eta\sigma^2}{4(b-d\theta)}$. Consequently, the third party's pricing algorithm is constructed to maximize the incremental profit from adoption, $\pi^{A}(\phi) - \pi^{NA}(\phi)$, and not an adopter's profit, $\pi^{A}(\phi)$. As long as the fee is less than the WTP (i.e., $\eta < 1$), it is possible for the third party and adopting firms to jointly be made better off. With that in mind, suppose the adopting firms approach the third party to form a hub-and-spoke cartel and propose to the third party that it redesign the pricing algorithm to maximize the adopting firms' profits. In exchange, the adopting firms will increase the fee from $\frac{\eta\sigma^2}{4(b-d\theta)}$ to $\frac{(1+\nu)\eta\sigma^2}{4(b-d\theta)}$ where $\nu \in \left(0, \frac{1-\eta}{\eta}\right)$.[19] It is then assumed that the design of the pricing algorithm does not affect the fee, though the fee is tied to parameters in a well-grounded way, given that it is a proportional increase over the equilibrium fee under competition.[20]

With the usual theory of collusion, collusive outcomes are sustained by the threat of some punishment—such as a temporary price war or permanent reversion to competition—in response to a firm's noncompliance. As our model has many firms, such a scheme would seem impractical because of its fragility, for all it would take is one firm out of many to deviate and cause a market-wide punishment. Consequently, the scheme considered is based on voluntary participation by the firms with no punishment for failing to join. Each firm decides whether or not to adopt the third party's collusive pricing algorithm or instead price on its own. Typically, such a scheme would not work if a firm is small, as it would prefer to be outside the cartel, undercutting the collusive price rather than inside, charging it. What is unique in the market being modeled is that failure to participate means foregoing the services of the third party. That a third party's pricing algorithm allows a firm to condition price on the demand state will prove integral to cartel stability and yield some novel implications of collusion.

Formally, the third party's problem is:

$$\max_{\phi} \pi^A(\phi) \text{ s.t. } \pi^A(\phi) - f \geq \pi^{NA}(\phi), \tag{8}$$

where

$$f = \frac{(1+v)\eta\sigma^2}{4(b-d\theta)}.$$

That is, the third party designs the pricing algorithm to maximize adopting firms' profits, subject to inducing each of those firms to adopt the pricing algorithm. Each firm pays a fee of $\frac{(1+v)\eta\sigma^2}{4(b-d\theta)}$.

The collusive equilibrium involves the third party solving (8) and the non-adopters choosing an optimal uniform price. It is shown in the Appendix that an equilibrium pricing algorithm is affine in the demand state.[21] Thus, the collusive equilibrium $(\hat{\alpha}, \hat{\gamma}, \hat{p})$ is defined by:

$$\begin{aligned}(\hat{\alpha}, \hat{\gamma}) = \arg\max_{\alpha,\gamma} \int (\alpha + \gamma a - c) \\ \times \big(a - (b - d\theta)(\alpha + \gamma a) + d(1-\theta)\hat{p}\big) G'(a) da \end{aligned} \tag{9}$$

$$\begin{aligned}\text{s.t.} \int (\alpha + \gamma a - c)(a - (b - d\theta)(\alpha + \gamma a) \\ + d(1-\theta)\hat{p}) G'(a) da - f \\ \geq \int (\hat{p} - c)\big(a - b\hat{p} + d\big(\theta(\alpha + \gamma a) + (1-\theta)\hat{p}\big)\big) G'(a) da \end{aligned} \tag{10}$$

$$\begin{aligned}\hat{p} = \arg\max_{p} \int (p - c)\big(a - bp + d\big(\theta(\hat{\alpha} + \hat{\gamma} a) \\ + (1-\theta)\hat{p}\big)\big) G'(a) da. \end{aligned} \tag{11}$$

By comparison, Harrington [20] examines the unconstrained problem of (9) and (11). As we will see, the addition of the participation constraint (10) significantly complicates the analysis, but also introduces an economic force that delivers some novel findings.

4 | Collusive Equilibrium

4.1 | Uniqueness and Characterization

Theorem 1 characterizes the third party's pricing algorithm when there is a collusive agreement. $\tilde{\alpha}$ in Equation (13) is the value of α (which is the intercept of the pricing algorithm) such that the constraint (10) binds, while α^U in (14) is the unconstrained solution to (9) and (11). Note that the first term in the expressions for $\tilde{\alpha}$ and α^U is α^I, which is the intercept for the equilibrium pricing algorithm in the absence of a collusive agreement, as reported in Equation (7). All proofs are in the Appendix.

Theorem 1. *There is a unique solution $(\hat{\alpha}, \hat{\gamma}, \hat{p})$ to (9–11) which is*

$$\hat{\alpha} = \min\{\tilde{\alpha}, \alpha^U\},\ \hat{\gamma} = \frac{1}{2(b-d\theta)}, \tag{12}$$

where

$$\begin{aligned}\tilde{\alpha} \equiv \frac{2b(b-d\theta)c + d\mu(1-2\theta)}{2(2b-d)(b-d\theta)} \\ + \left(\frac{(2b - d(1-\theta))\sqrt{1-(1+v)\eta}}{2(2b-d)\sqrt{b^2 - bd\theta}}\right)\sigma, \end{aligned} \tag{13}$$

$$\begin{aligned}\alpha^U \equiv \frac{2b(b-d\theta)c + d\mu(1-2\theta)}{2(b-d\theta)(2b-d)} \\ + \frac{d\theta(2b - d(1-\theta))(\mu - (b-d)c)}{(2b-d)\big(2b(2b - d(1+\theta)) + d^2\theta(1-\theta)\big)}, \end{aligned} \tag{14}$$

and

$$\hat{p} = \frac{\mu + bc + d\theta(\hat{\alpha} + \hat{\gamma}\mu)}{2b - d(1-\theta)}. \tag{15}$$

The slope of the pricing algorithm under collusion is $\frac{1}{2(b-d\theta)}$ which is the same as under competition, see Equation (7), and the same as that which maximizes the joint profits of the θ firms (ignoring the participation constraint). In other words, a third party always designs its pricing algorithm so that the price responds to a change in the demand state so as to maximize adopting firms' joint profits. To understand this finding, decompose the third party's problem into two subproblems: (1) find the optimal average price for the pricing algorithm; and (2) among those pricing algorithms with that average price, find the optimal pricing algorithm. It is shown in the Appendix (see Lemma A1) that the solution to the second subproblem is $\gamma = \frac{1}{2(b-d\theta)}$, and this is true for any average price. Given $\gamma = \frac{1}{2(b-d\theta)}$, α is then chosen to solve the first subproblem. Next, note that the non-adopter's profit depends only on the average price of the pricing algorithm. Thus, holding the average price fixed, the non-adopter's profit is constant with respect to the pricing algorithm, which means $\gamma = \frac{1}{2(b-d\theta)}$ not only maximizes an adopter's profit but also maximizes the WTP, which is the difference between the adopter's profit and the non-adopter's profit. In sum, the pricing algorithm responds to the demand state in the same way under competition and collusion.[22] Turning to the intercept of the collusive pricing algorithm, it is the unconstrained value α^U when $\alpha^U \leq \tilde{\alpha}$ so the constraint is satisfied, and otherwise is $\tilde{\alpha}$ which is the highest value satisfying the constraint.

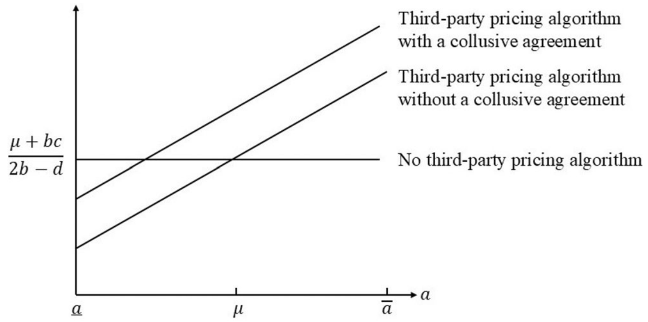

FIGURE 1 | Pricing algorithms.

The effect of a third party and a collusive agreement on the pricing algorithm is depicted in Figure 1. In the absence of a third party, firms set a uniform price $\frac{\mu+bc}{2b-d}$. Introducing a third party means that a firm's pricing algorithm now responds to the demand state, though, in the absence of any collusive agreement, there is no supracompetitive markup as the average price is still $\frac{\mu+bc}{2b-d}$. When there is a collusive agreement, the pricing algorithm has a price response to demand changes in the same manner (i.e., the slope is unchanged), but the pricing algorithm is shifted up, so the average price is higher.

4.2 | Conditions Under Which the Participation Constraint Binds

It is useful to think of the third party as facing two constraints when designing its pricing algorithm. Both of these constraints come from the prospect of a firm not adopting the pricing algorithm and instead optimizing against the supracompetitive average price of the adopting firms. The first (explicit and hard) constraint has been discussed, which is that a firm will only adopt if it is more profitable than not adopting. Referred to as the *cartel participation constraint*, it is reflected in $\widetilde{\alpha}$. While that constraint applies to the θ firms who adopt, the other constraint is due to the $1-\theta$ firms who do not. This second (implicit and soft) constraint is analogous to the one faced by any non-all-inclusive cartel, which is that non-colluding firms will undercut the collusive price and siphon off demand. Here, it is the non-adopting firms that play this role and cause the third party to restrain the size of the supracompetitive markup built into the pricing algorithm. Referred to as the *competitive fringe constraint*, it is reflected in α^U.

Let us now explore how two key parameters—the adoption rate θ and the demand variance σ^2 (which drives the third party's efficiency)—influence which of these constraints is determinative of the collusive price.[23] In deciding whether to adopt the pricing algorithm, a firm faces the following trade-off. Adoption allows it to condition the price on the demand state, and that benefit is increasing in the variability of demand, σ^2. Not adopting allows it to choose a uniform price that undercuts the high average price of rival firms who are adopting the pricing algorithm, and that benefit is increasing in the pricing algorithm's supracompetitive markup. When σ^2 is low, the third party is forced to limit the markup to induce firms to adopt the pricing algorithm. Hence, the cartel participation constraint is binding—as reflected in $\widetilde{\alpha} < \alpha^U$—and $\widehat{\alpha} = \widetilde{\alpha}$. When instead σ^2 is high, the third party can set the unconstrained optimum and firms will still adopt the pricing algorithm. The cartel participation constraint is not binding, so $\widetilde{\alpha} > \alpha^U$ and $\widehat{\alpha} = \alpha^U$, and it is the competitive fringe constraint which is determinative of the markup.

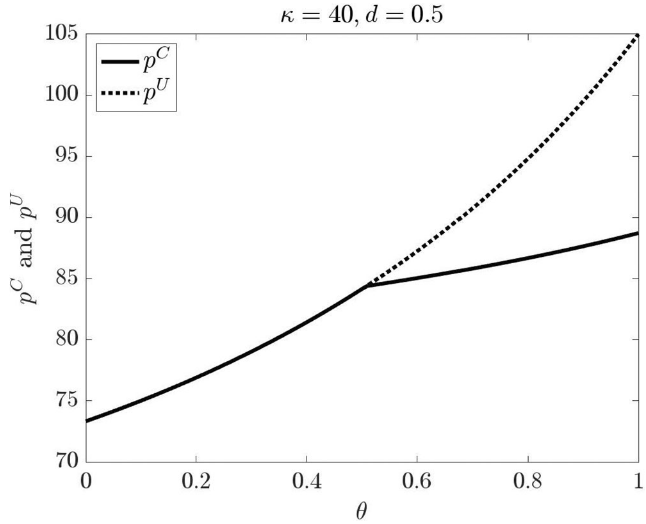

FIGURE 2 | Effect of adoption rate.

Theorem 2. *There exists* $\widehat{\sigma}^2 > 0$ *such that*

$$\widehat{\alpha} = \begin{cases} \widetilde{\alpha} & if\, \sigma^2 \in \left(0, \widehat{\sigma}^2\right] \\ \alpha^U & if\, \sigma^2 \geq \widehat{\sigma}^2 \end{cases}.$$

Next, let us turn to the adoption rate. When the adoption rate is low, the presence of many non-adopting firms undercutting the adopting firm's supra-competitive average price means the third party will restrain its magnitude. Consequently, α^U is low which implies $\alpha^U < \widetilde{\alpha}$ so it is the competitive fringe constraint which is determinative, thereby resulting in $\widehat{\alpha} = \alpha^U$. When instead, the adoption rate is high, the third party would like to be able to set a high supracompetitive price—because there are few non-adopting firms to undercut it—but it is constrained by having to induce firms to adopt the pricing algorithm. Hence, $\widetilde{\alpha} < \alpha^U$ so the cartel participation constraint binds and, therefore, $\widehat{\alpha} = \widetilde{\alpha}$. Consistent with that intuition, Theorem 3 proves this property when the adoption rate is sufficiently low or sufficiently high.[24]

Theorem 3. *If* $\sigma^2 < \frac{d^2(\mu-(b-d)c)^2}{4b(b-d)(1-(1+\upsilon)\eta)}$ *then there exists* $\varepsilon > 0$ *such that*

$$\widehat{\alpha} = \begin{cases} \alpha^U & if\, \theta \in [0, \varepsilon] \\ \widetilde{\alpha} & if\, \theta \in [1-\varepsilon, 1] \end{cases}.$$

Building on this finding, numerical analysis shows there is a threshold adoption rate such that the cartel participation constraint binds if and only if the adoption rate exceeds that threshold. This property is depicted in Figure 2 where the average collusive price, $p^C = \widehat{\alpha} + \widehat{\gamma}\mu$, equals the average unconstrained collusive price, $p^U = \alpha^U + \widehat{\gamma}\mu$, until the adoption rate reaches some threshold value, after which $p^C < p^U$.[25]

5 | Comparative Statics

The average supracompetitive markup is defined as the difference between the average adopter's price, $\hat{\alpha} + \hat{\gamma}\mu$, and the competitive price, $\frac{\mu+bc}{2b-d}$. As one might expect, the higher is the adoption rate, so more firms are members of the hub-and-spoke cartel, the higher the cartel overcharge.

Theorem 4. *The average collusive price,*

$$\hat{\alpha} + \hat{\gamma}\mu = \min\left\{\tilde{\alpha} + \frac{\mu}{2(b-d\theta)}, \alpha^U + \frac{\mu}{2(b-d\theta)}\right\},$$

and the average supracompetitive markup,

$$\min\left\{\tilde{\alpha} + \frac{\mu}{2(b-d\theta)}, \alpha^U + \frac{\mu}{2(b-d\theta)}\right\} - \frac{\mu+bc}{2b-d},$$

are increasing in the adoption rate θ.

As non-adopting firms optimize their uniform price in response to the average collusive price, it follows from prices being strategic complements and from Theorem 4 that a non-adopter's price and markup are also increasing in the adoption rate. Commonly referred to as the umbrella effect, a non-adopter's supracompetitive markup is attributable to the collusive agreement among adopting firms.

A novel finding is that the average collusive price depends on the efficiency delivered by the third party. Specifically, the greater the demand variance, the higher the supracompetitive markup from collusion.

Theorem 5. *There exists $\hat{\sigma}^2 > 0$ such that if $\sigma^2 \in (0, \hat{\sigma}^2]$ then the average collusive price and supracompetitive markup are increasing in σ^2 and decreasing in η and υ; and if $\sigma^2 > \hat{\sigma}^2$ then they are independent of σ^2, η, and υ.*

When σ^2 is not too high, so the cartel participation constraint binds, then the supracompetitive markup is

$$\left(\frac{(2b - d(1-\theta))\sqrt{1-(1+\upsilon)\eta}}{2(2b-d)\sqrt{b^2 - bd\theta}}\right)\sigma,$$

from which it is easy to see that it is increasing in the demand variance. In building a supracompetitive markup into its pricing algorithm, the third party makes it more attractive for a firm not to adopt the pricing algorithm and instead undercut the average collusive price with its uniform price. However, nonadoption means foregoing the efficiency of being able to condition price on the demand state. Hence, as the efficiency is greater, the third party can set a higher markup and still induce firms to adopt the pricing algorithm.

That more demand variability results in higher average collusive prices has also been shown in Sugaya and Wolitzky [19] so it is proving to be a robust finding. It holds for the setting of this paper, which models a firm's decision to adopt the third party's pricing algorithm while assuming an adopting firm implements the algorithm's generated prices, and the setting in Sugaya and Wolitzky [19], which models an adopting firm's decision to implement the algorithm's generated prices while assuming a firm adopts the third party's pricing algorithm. This result is likely to be general because it is driven by the need to incentivize firms to adopt the pricing algorithm, which is made easier when the third party delivers more value.

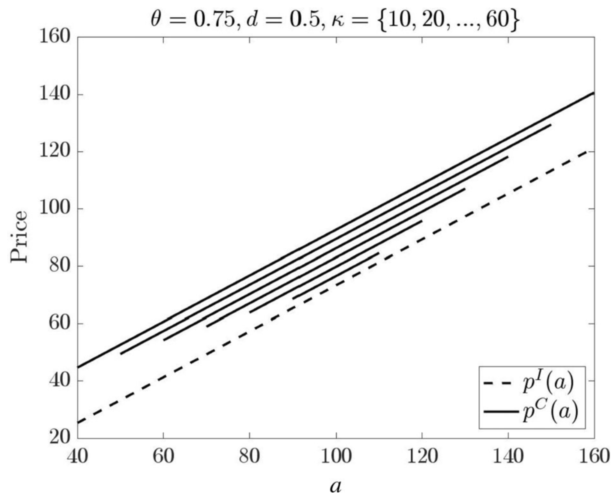

FIGURE 3 | Effect of demand variability.

While we have focused on how the third party's efficiency influences the supracompetitive markup, it is also affected by the fee. Indeed, it is the net surplus—a firm's value from conditioning price on the demand state (i.e., its WTP) less the price it has to pay the third party for that service—which a firm loses by opting out of the collusive agreement; thereby giving the third party leverage when setting the pricing algorithm's markup. η and υ are parameters influencing the fee paid to the third party. In the absence of a collusive agreement, η is the fraction of the surplus captured by the third party, while υ is in the proportional increase in the third party's fee associated with it participating in a collusive agreement. The more effective the third party is in extracting surplus—as reflected in a higher value for η or υ—the smaller is a firm's net surplus, which then reduces the magnitude of the supracompetitive markup that can be sustained. That less upstream supplier bargaining power facilitates hub-and-spoke collusion is consistent with that shown in Garrod et al. [5] for the case of a more standard input (rather than a pricing algorithm), though the economic forces are quite different.[26]

To illustrate the effect of a third party's efficiency on collusive pricing, Figure 3 provides a numerical example with the demand state a having a uniform distribution over $[100-\kappa, 100+\kappa]$ so $\sigma^2 = \kappa^2/3$. The third party's pricing algorithm is plotted without a collusive agreement (dashed line) and with a collusive agreement (solid lines) for $\kappa \in \{10, 20, \ldots, 60\}$.[27] A higher value for κ means a wider range of possible demand states, a higher demand variance, and a greater efficiency delivered by the third party. The collusive pricing algorithm shifts up when κ is higher, so a more efficient third party designs the pricing algorithm to generate a higher price for each demand state. We find that a third party's efficiency is a facilitating factor for collusion.

In summary, when there is a hub-and-spoke collusive agreement, the average collusive price is increasing in the adoption rate and the demand variance. In contrast, the analysis in Section 2.2 showed that, under competition, a third party's pricing algorithm results in the average price being independent of the adoption rate and demand variance. Thus, evidence that firms subscribing to a third party's services have a higher average price when more firms adopt, and when operating in markets with more variable demand supports the presence of collusion.

6 | Some Implications for Competition Law and Policy

In this section, I discuss some practical implications of the theory for competition law and policy. Traditionally, in the United States and increasingly in other jurisdictions such as the European Union, customer damages are a crucial financial penalty to be imposed on convicted cartel members. Section 6.1 shows how the theory can be used to assist in estimating customer damages. A novel finding of the theory is that a data analytics company's efficiency is a facilitating factor for collusion, and Section 6.2 discusses how this creates some challenges for competition policy.

6.1 | Estimating the Collusive Overcharge

The standard formula for customer damages is the overcharge multiplied by the number of units sold, where the overcharge is the price that was paid less the price that would have been paid but for the collusive agreement. The primary challenge in damage estimation is coming up with credible but-for (or counterfactual) prices. Here, I describe how our theory of third-party pricing algorithms can assist with this task.[28]

The first point to make is that the proper counterfactual is not firms' pricing without the assistance of the third party, but rather firms' pricing with the third party's assistance, without a collusive agreement. Hence, the overcharge is $\left(\hat{\alpha}+\hat{\gamma}a\right)-\left(\alpha^I+\gamma^I a\right)$ where the third party's pricing algorithm is $\hat{\alpha}+\hat{\gamma}a$ with a collusive agreement (see Section 4.1) and $\alpha^I+\gamma^I a$ without a collusive agreement (see Section 2.2). As depicted in Figure 1, the overcharge is constant with respect to the demand state and equals

$$\left(\frac{(2b-d(1-\theta))\sqrt{1-(1+\upsilon)\eta}}{2(2b-d)\sqrt{b^2-bd\theta}}\right)\sigma,$$

when σ^2 is sufficiently low (so the participation constraint binds) and

$$\frac{d\theta(2b-d(1-\theta))(\mu-(b-d)c)}{(2b-d)(2b(2b-d(1+\theta))+d^2\theta(1-\theta))},$$

when σ^2 is sufficiently high (so the participation constraint does not bind). If, instead, one were to incorrectly take the counterfactual to be pricing without a third party—which is the constant price shown in Figure 1—the misestimated overcharge would increase in the demand state and could be negative when demand is weak.[29]

If only data from the cartel period is available, then adopting firms' prices with the collusive agreement are observed, but adopting firms' prices without the collusive agreement are not. Our theory can be useful here. Assuming adoption is less than complete, non-adopting firms' prices are observed. From Section 2.2, we know that, in the but-for scenario of no collusive agreement, the average price of non-adopting firms is the same as the average price of adopting firms. Subject to a caveat, we can then use the observed average price of non-adopting firms as a proxy for the but-for average price of adopting firms. Using price data from the cartel period, the theory supports measuring the overcharge as the average price of adopting firms minus the average price of non-adopting firms.[30]

The caveat is the umbrella effect. Generally, non-cartel members' prices exceed but-for levels because they will price higher in response to the supracompetitive prices of cartel members. Analogously, non-adopting firms' prices exceed their but-for levels due to optimally responding to the higher average prices of adopting firms (when there is collusion). Using non-adopting firms' prices as a proxy, the expectation of the estimated overcharge is

$$\hat{\alpha}+\hat{\gamma}\mu-\frac{\mu+bc+d\theta\left(\hat{\alpha}+\hat{\gamma}\mu\right)}{2b-d(1-\theta)}=\frac{(2b-d)\left(\hat{\alpha}+\hat{\gamma}\mu\right)-(\mu+bc)}{2b-d(1-\theta)}, \tag{16}$$

where the expected prices of adopters is $\hat{\alpha}+\hat{\gamma}\mu$ and nonadopters is $\frac{\mu+bc+d\theta(\hat{\alpha}+\hat{\gamma}\mu)}{2b-d(1-\theta)}$. Given the expectation of the true overcharge is

$$\hat{\alpha}+\hat{\gamma}\mu-\frac{\mu+bc}{2b-d}, \tag{17}$$

the underestimate due to the umbrella effect is obtained by subtracting Equation (16) from Equation (17), which yields

$$\frac{d\theta\left(\hat{\alpha}+\hat{\gamma}\mu-\frac{\mu+bc}{2b-d}\right)}{2b-d(1-\theta)}.$$

Bias as a percentage of the true overcharge is

$$\frac{d\theta}{2b-d(1-\theta)}.$$

Given the bias, this method provides a lower bound on the overcharge.

Another caveat is that there could be selection bias if the firms that choose to adopt the pricing algorithm differ in ways that also affect how they price. For example, if, for some reason, adopting firms have a higher cost than non-adopting firms, then the average price of non-adopting firms will be a downward-biased estimate of the average price of adopting firms and, therefore, the overcharge will be overestimated. It is difficult to say in what direction this bias will go without a better understanding of what drives firms' adoption decisions.[31]

In closing, let me briefly mention that it may be possible to learn the third party's but-for pricing algorithm and use it to estimate overcharges. If the third party stores past pricing algorithms, its pricing algorithm prior to the collusive agreement may be obtainable through legal discovery. Second, using pre-cartel data, it may be possible to estimate the but-for pricing algorithm. With both methods, one would need to learn how price depends on the demand state and the adoption rate because, to produce but-for prices, the but-for pricing algorithm has to generate prices for the demand states and adoption rates that occurred during the cartel period. That price series, along with adopting firms' observed prices, would then yield an estimate of overcharges.

6.2 | A Conundrum for the Enforcement of Competition Law

The third party delivers an efficiency in the sense that it allows a firm to better solve the problem of setting the best price in response to market conditions. This deliverable is a legitimate one, even though it may or may not be beneficial to consumers. A novel finding of our analysis is that this efficiency is a facilitating factor for collusion. Markets with more variable demand provide greater scope for a data analytics company to deliver value to firms, and that higher value allows it to program in a higher supracompetitive markup when there is a collusive agreement. Here, we discuss some implications related to the enforcement of competition law.[32]

The challenge of third-party pricing algorithms for competition policy is well-recognized. Whenever competitors allow a common agent to influence their prices, there is the risk that independent price-setting will be displaced by coordinated pricing. At the same time, there is a legitimate basis for this relationship when the common agent is a data analytics company. Consequently, intervention to address the risk of anticompetitive harm needs to be tempered by the risk of interfering with the efficiencies provided by data analytics. This challenge has been noted by Ezrachi and Stucke [1], Weber [31], and Ezrielev [32] among other commentators.

The point to be made is that this enforcement challenge is more severe than originally appreciated. It is not simply that efficiencies *could* occur along with the risk of anticompetitive harm, but rather, as the theory here predicts, they *will* occur. It is exactly markets where efficiencies are strongest—so intervention by a competition authority or private litigants could be socially costly—that the risk of anticompetitive harm is greatest, so failure to intervene could be socially costly. At a minimum, it implies that a policy of only investigating markets when prices are high (so concerns about harm are acute) and demand variability is low (so concerns about interfering with efficiency are limited) is inadequate because the greatest risk for harm is when demand variability is high.

When collusion tends to occur with efficiencies, there can be a justification for not having a per se or by object prohibition against collusion. A case in point is resale price maintenance (RPM). Though it is a by-object violation of Article 101 of the TFEU, it has been evaluated using the rule of reason in the United States since a Supreme Court ruling in 2007.[33] The reason is that unconstrained price competition can stifle non-price competition. An agreement among retailers to charge higher prices can energize non-price competition—such as with respect to customer service—to the point that consumers are better off. However, this line of reasoning is not applicable to third-party pricing algorithms because the causality runs in the other direction. While price collusion facilitates an efficiency in the context of RPM, it is the efficiency that facilitates price collusion in the context of a data analytics company supplying a pricing algorithm. As the efficiency can be delivered without a collusive agreement, a per se or by object prohibition remains appropriate even when a data analytics company is delivering an efficiency along with coordinating prices.[34]

7 | Concluding Remarks

In supplying a pricing algorithm, a data analytics company's deliverable is assisting firms in setting profit-maximizing prices. For example, a third party's pricing algorithm may be better able to adjust price to a changing demand state. At the same time, whenever competitors are being assisted by the same third party, there is the concern that the third party's deliverables will enter into anticompetitive territory. In the U.S. markets for apartments and hotel rooms, a data analytics company and its subscribing firms have been accused of forming a hub-and-spoke cartel. The objective of this paper has been to explore the form and effect of hub-and-spoke collusion in this class of markets.

Our analysis uncovered a novel determinant of collusive conduct. A challenge to collusion in this setting is inducing firms to adopt a pricing algorithm with a supracompetitive markup. A firm may prefer not to adopt and instead set a price that undercuts the average supracompetitive price of adopting firms. The downside to not adopting is that a firm foregoes the third party's efficiency of conditioning price on the demand state. We found that it is this efficiency that determines the profitability of collusion. The higher the demand variance, the more firms value the third party's services and, consequently, the more willing they are to adopt a pricing algorithm with a supracompetitive markup. The third party's efficiency is a facilitating factor for collusion.

Acknowledgments

I gratefully acknowledge the research assistance of Vivian Zhang and the comments of two anonymous referees, Jessica Jeffers, Nate Miller, Alex Wolitzky, and participants at seminars at the Universität Mannheim, Universitat Pompeu Fabra, Universitat de Barcelona, and Universität Wien, and at the 2024 DICE Competition Policy Conference, 2025 International Industrial Organization Conference, and 2025 CRESSE Conference. I had been retained by a defendant in private litigation associated with the use of a data analytics company. This study is not funded in whole or in part by any person or entity, either directly or indirectly, related to that litigation. No client or other interested party has a right to review, or has reviewed, this paper.

Endnotes

[1] Bason et al. v. RealPage, Inc., et al., No. 3:22-cv-01611-WQH-MDD, U.S. District Court, Southern District of California, October 18, 2022, para. 5.

[2] Hanson Dai et al. v. SAS Institute, Inc., et al., No. 3:24-cv-02537, U.S. District Court, Northern District of California, April 26, 2024, para. 9.

[3] "If a sufficiently large proportion of an industry uses a single algorithm to set prices, this could result in a ... structure that may have the ability and incentive to increase prices." United Kingdom Competition & Markets Authority [2, para. 5.21].

[4] A third party "knows or accepts [it] could contribute to a collusive market outcome [and] it is even conceivable that [they] see such a contribution as an advantage, as it makes the algorithm more attractive for users interested in profit maximization." German Monopolies Commission [3, para. 263].

[5] "[C]oncerns of coordination would arise if firms outsourced the creation of algorithms to the same IT companies and programmers [and] [t]his might create a sort of 'hub and spoke' scenario where co-ordination is, willingly or not, caused by competitors using the same 'hub' for developing their pricing algorithms and end up relying on the same algorithms." OECD [4, para 68].

[6] The U.S. Senate bill is the "Preventing Algorithmic Collusion Act" which was put forward in February 2024 and reintroduced in January 2025. During 2024–2025, laws targeted at the apartment rentals market were passed in Berkeley, Minneapolis, Philadelphia, and San Francisco, and the states of California and New York.

[7] For example, it is shown in Bos and Harrington [8] that firms with low capacity will choose not to join a cartel.

[8] An exception is Ichihashi [17], which models strategic uncertainty rather than demand uncertainty.

[9] The model and analysis of this section borrows from Harrington [20].

[10] This demand specification is an extension of a common specification—see, for example, Vives [28], p. 146—to when there is a continuum of firms. When there is a finite number of firms, a firm's decision to adopt the third party's pricing algorithm affects the adoption rate and, as we'll see, affects the design of the pricing algorithm. With many firms, that effect is small, and it greatly simplifies the analysis by assuming the effect is zero, which is the case with a continuum of firms.

[11] This assumption can be grounded in bargaining between the third party and a firm. Denote π^A and π^{NA} as a firm's profit from adopting and not adopting the pricing algorithm, respectively, so the WTP is $\pi^A - \pi^{NA}$. If the third party and a firm come to an agreement, then the firm's payoff is $\pi^A - f$ and the third party's payoff is f. If they fail to come to an agreement, then the firm's payoff is π^{NA}, and the third party's payoff is zero. The Nash Bargaining Solution fee is:

$$\eta\left(\pi^A - \pi^{NA}\right) = \arg\max_f f^\eta\left(\pi^A - f - \pi^{NA}\right)^{1-\eta},$$

where $\eta \in (0,1)$ measures the bargaining power of the third party. As assumed, the fee is some fraction of the WTP.

[12] An alternative description of this assumption is that there is a firm-specific adoption cost (in addition to paying the fee) which is zero for θ firms and very large (exceeding the WTP) for $1-\theta$ firms. For the competitive setting, Harrington [7] allows for a continuous distribution over adoption costs and shows that the qualitative properties of the pricing algorithm are robust. I conjecture the same is true for the collusive setting.

[13] θ is the fraction of firms that have the option to adopt the third party's pricing algorithm, which is exogenous. In equilibrium, all firms with that option will adopt, so θ is also the adoption rate, which is endogenous. I will refer to conducting comparative statics with respect to the adoption rate, even though, literally, it is with respect to the fraction of firms with the option to adopt.

[14] Currently, there is no theory that both endogenizes the decision to adopt a third party's pricing algorithm and, having adopted it, the decision to implement the algorithm's generated prices. Harrington [6, 7, 20] only endogenizes the adoption decision, while Hickok [18] and Sugaya and Wolitzky [19] only endogenizes the implementation decision.

[15] As used in Harrington [20], the superscript I refers to when firms's conduct is independent (or competitive) as opposed to when it is coordinated (or collusive).

[16] Assumed is a simultaneous-move structure in that the third party chooses its pricing algorithm given its conjecture of non-adopting firms' prices, and a non-adopting firm chooses its price given its conjecture of the third party's pricing algorithm (and other firms' adoption decisions) and other non-adopting firms' prices. Alternatively, a sequential-move structure could be assumed in which the third party's pricing algorithm is observed by the non-adopting firms prior to them choosing their prices. I do not believe the qualitative results would change. I chose to abstract from any commitment effect associated with using a pricing algorithm to focus on a third party's incentives when it comes to designing it.

[17] That the solution is affine can be shown through pointwise optimization whereby an optimal price is chosen for each demand state. The constructed pricing function is affine in the demand state.

[18] For a more detailed explanation, see p. 282–284 in Harrington [20].

[19] If ν were to exceed $\frac{1-\eta}{\eta}$, then we will find the adopting firms are worse off as the fee hike exceeds the rise in incremental profit from the redesigned pricing algorithm.

[20] Assuming the design of the pricing algorithm does not affect the fee greatly enhances tractability and is a modeling simplification also used in Sugaya and Wolitzky [19]. Their model does not have a fee, as it is assumed firms adopt the third party's pricing algorithm, and if one supposes there is an implicit fee, it is fixed and independent of the design of the pricing algorithm.

[21] After the proof, I explain why the pricing algorithm is affine in my model but not in the model of Sugaya and Wolitzky [19].

[22] More of an economic explanation as to why $\gamma = \frac{1}{2(b-d\theta)}$ under competition is provided in section 4 of Harrington [20].

[23] As a reminder, θ is an exogenous parameter which is the fraction of firms who have the option to adopt the third party's pricing algorithm. In equilibrium, all θ firms adopt, so it is also the adoption rate.

[24] There is an upper bound on σ^2 because if σ^2 is sufficiently large, then the solution is not constrained, even when the adoption rate is 1, because the efficiency is so valuable.

[25] Figure 2 is based on the following parameterization. $b = 1, d = 0.50, c = 10$, the demand state a has a uniform distribution over $[\mu - \kappa, \mu + \kappa]$ so $\sigma^2 = \kappa^2/3$, mean $\mu = 100$ and $(1+\upsilon)\eta = 1/2$ so the fee is half of the WTP. Finally, $\theta \in \{0, 0.01, \ldots, 0.99, 1\}$. Figure 2 shows results for $\kappa = 40$ and similar findings have been found for $\kappa \in \{20, 60, 80\}$ and $d \in \{0.25, 0.75\}$.

[26] For an explanation, see section 3.5 of Garrod et al. [5].

[27] The other parameters are $b = 1, d = 0.50, c = 10, \theta = 0.75$, and $(1+\upsilon)\eta = 1/2$ so the fee is half of the WTP.

[28] For additional discussion on estimating damages associated with a third party's pricing algorithm, see Harrington [20] and Kheyfets and Kully [29].

[29] This conclusion is not dependent on a firm's pricing algorithm without the third party being a fixed price. Suppose instead a firm could condition price on the demand state, but did so less effectively than the third party, so that its price was less sensitive to the demand state; in other words, the slope is flatter than the collusive pricing algorithm in Figure 1. It would then still be true that the misestimated overcharge is increasing in the demand state and could be negative for weak demand states.

[30] This method is a familiar one: During the cartel period, compare the prices of cartel members with the prices of non-cartel members. However, it is only because it was proven that the average but-for price of adopting firms is the same as the average but-for price of nonadopting firms that one can use non-adopting firms' prices as a proxy for adopting firms' prices when there is no collusive agreement.

[31] While such selection bias is possible, but it need not occur. Harrington [7] allows firms to differ in a firm-specific adoption cost, which affects their adoption decision but not pricing. In that case, there is no selection bias.

[32] For a broader discussion of some of the challenges of third-party pricing algorithms for competition policy, see Harrington [30].

[33] *Leegin Creative Leather Products, Inc. v. PSKS, Inc.*, 551 U.S. 877 (2007).

[34] As a reminder, this is an efficiency from the perspective of the firm. It may or may not be a procompetitive efficiency because it may or may not benefit consumers.

[35] See, e.g., Riesz and Sz.-Nagy [33].

[36] The derivation is available on request.

References

1. A. Ezrachi and M. E. Stucke, “Artificial Intelligence and Collusion: When Computers Inhibit Competition,” *University of Illinois Law Review* 2017 (2017): 1775–1810.

2. United Kingdom Competition & Markets Authority, “Pricing Algorithms: Economic Working Paper on the Use of Algorithms to Facilitate Collusion and Personalised Pricing,” 2018.

3. German Monopolies Commission, “Algorithms and Collusion,” in *XXII. Biennial Report* (German Monopolies Commission, 2018).

4. OECD, “Algorithms and Collusion - Background Note by the Secretariat,” in *DAF/Comp(2017)*, vol. 4 (OECD, 2017).

5. L. Garrod, J. E. Harrington, and M. Olczak, *Hub-and-Spoke Cartels: Why They Form, How They Operate, and How to Prosecute Them* (MIT Press, 2021).

6. J. E. Harrington, Jr., “The Effect of Outsourcing Pricing Algorithms on Market Competition,” *Management Science* 68 (2022): 6889–6906.

7. J. E. Harrington, Jr., “The Effect of Demand Variability on the Adoption and Design of a Third Party’s Pricing Algorithm,” *Economics Letters* 244 (2024): 112011.

8. I. Bos and J. E. Harrington, Jr., “Endogenous Cartel Formation With Heterogeneous Firms,” *RAND Journal of Economics* 41 (2010): 92–117.

9. E. R. Arunachaleswaran, N. Collina, S. Kannan, A. Roth, and J. Ziani, “Algorithmic Collusion Without Threats,” 2024 Working Paper.

10. Z. Y. Brown and A. MacKay, “Competition in Pricing Algorithms,” *American Economic Journal: Microeconomics* 15 (2023): 109–156.

11. Z. Y. Brown and A. MacKay, “Algorithmic Coercion With Faster Pricing,” 2025 Working Paper.

12. R. Lamba and S. Zhuk, “Pricing With Algorithms,” 2024 Working Paper.

13. M. Leisten, “Algorithmic Competition, With Humans,” 2024 Working Paper.

14. B. Salcedo, “Pricing Algorithms With Tacit Collusion,” 2015 Working Paper.

15. L. Musolff, “Algorithmic Pricing Facilitates Tacit Collusion: Evidence From E-Commerce,” 2024 Working Paper.

16. J. Johnson, A. Rhodes, and M. Wildenbeest, “Platform Design When Sellers Use Pricing Algorithms,” *Econometrica* 91 (2023): 1841–1879.

17. S. Ichihashi, “Platform-Enabled Algorithmic Pricing,” 2025 Working Paper.

18. N. Hickok, “Algorithm Design Meets Information Design: Price Recommendation Algorithms on Online Platforms,” 2024 Working Paper.

19. T. Sugaya and A. Wolitzky, “Collusion With Optimal Information Disclosure,” 2025 Working Paper.

20. J. E. Harrington, Jr., “An Economic Test for an Unlawful Agreement to Adopt a Third-Party’s Pricing Algorithm,” *Economic Policy - Special Issue on Artificial Intelligence and the Economy* 40 (2025): 263–295.

21. S. Assad, R. Clark, D. Ershov, and L. Xu, “Algorithmic Pricing and Competition: Empirical Evidence From the German Retail Gasoline Market,” *Journal of Political Economy* 132 (2024): 723–771.

22. S. Calder-Wang and G. H. Kim, “Algorithmic Pricing in Multifamily Rentals: Efficiency Gains or Price Coordination?,” 2024. Working Paper.

23. N. Sahuguet and A. Walckiers, “A Theory of Hub-and-Spoke Collusion,” *International Journal of Industrial Organization* 53 (2017): 353–370.

24. J. Do, “Information Exchange in Hub-and-Spoke Cartels,” 2023 Working Paper.

25. J. Miklós-Thal and C. Tucker, “Collusion by Algorithm: Does Better Demand Prediction Facilitate Coordination Between Sellers?,” *Management Science* 4 (2019): 1552–1561.

26. J. O’Connor and N. E. Wilson, “Reduced Demand Uncertainty and the Sustainability of Collusion: How AI Could Affect Collusion,” *Information Economics and Policy* 54 (2021): 100882.

27. S. Martin and A. Rasch, “Demand Forecasting, Signal Precision, and Collusion With Hidden Actions,” *International Journal of Industrial Organization* 24 (2024): 103036.

28. X. Vives, *Oligopoly Pricing: Old Ideas and New Tools* (MIT Press, 1999).

29. M. Kheyfets and D. Kully, “Antitrust Injury and Damages in Algorithmic Collusion Cases: ‘Another Frontier’?,” *Antitrust Source* 26 (2025): 1–14.

30. J. E. Harrington, Jr., “The Challenges of Third-Party Pricing Algorithms for Competition Law,” *Theoretical Inquiries in Law* 26 (2025): 123–145.

31. B. C. Weber, “Hub-and-Spoke Conspiracies: Can Big Data and Pricing Algorithms Form the Rim?,” *SMU Science and Technology Law Review* 26 (2023): 25–84.

32. J. Ezrielev, *DOJ’S War on Progress in Rental Markets* (CPI Columns, 2024).

33. F. Riesz and B. Sz.-Nagy, *Functional Analysis* (Dover Publications, 1990).

Appendix A

Proof that a Solution is Affine

The general form of the problem is:

$$\hat{\phi} = \arg\max_{\phi} \int (\phi(a) - c)(a - (b - d\theta)\phi(a) + d(1-\theta)\hat{p})G'(a)da \tag{A1}$$

$$\text{s.t.} \int (\phi(a) - c)\big(a - (b - d\theta)\phi(a) + d(1-\theta)\hat{p}\big)G'(a)da - f \geq \int (\hat{p} - c)\big(a - b\hat{p} + d\big(\theta\phi(a) + (1-\theta)\hat{p}\big)\big)G'(a)da$$

$$\hat{p} = \arg\max_{p} \int (p - c)\Big(a - bp + d\Big(\theta\hat{\phi}(a) + (1-\theta)\hat{p}\Big)\Big)G'(a)da.$$

Integrating gives us

$$\hat{\phi} = \arg\max_{\phi} \int (\phi(a) - c)\big(a - (b - d\theta)\phi(a) + d(1-\theta)\hat{p}\big)G'(a)da$$

$$s.t. \int (\phi(a) - c)\big(a - (b - d\theta)\phi(a) + d(1-\theta)\hat{p}\big)G'(a)da - f \geq (\hat{p} - c)\big(\mu - b\hat{p} + d\big(\theta E[\phi(a)] + (1-\theta)\hat{p}\big)\big)$$

$$\hat{p} = \arg\max_{p}(p - c)\Big(\mu - bp + d\Big(\theta E\Big[\hat{\phi}(a)\Big] + (1-\theta)\hat{p}\Big)\Big).$$

Note that the objective and the LHS of the constraint (except for f) are the same expression. Let us perform some manipulations of that expression:

$$\int (\phi(a) - c)\big(a - (b - d\theta)\phi(a) + d(1-\theta)\hat{p}\big)G'(a)da$$

$$= \int \phi(a)(a-(b-d\theta)\phi(a))G'(a)da$$
$$- \int c\big(a-(b-d\theta)\phi(a)+d(1-\theta)\hat{p}\big)G'(a)da$$
$$+ \int \phi(a)d(1-\theta)\hat{p}G'(a)da$$
$$= \int \phi(a)(a-(b-d\theta)\phi(a))G'(a)da$$
$$- c\big(\mu-(b-d\theta)E[\phi(a)]+d(1-\theta)\hat{p}\big)$$
$$+ E[\phi(a)]d(1-\theta)\hat{p}$$

$$= \int \phi(a)(a-(b-d\theta)\phi(a))G'(a)da$$
$$- c(\mu-(b-d\theta)E[\phi(a)]) + (E[\phi(a)]-c)d(1-\theta)\hat{p}. \tag{A2}$$

Perform some manipulations of the first term:

$$\int \phi(a)(a-(b-d\theta)\phi(a))G'(a)da$$
$$= E[\phi(a)a] - (b-d\theta)E\left[\phi(a)^2\right]$$
$$= \mathrm{Cov}[\phi(a)a] + E[\phi(a)]\mu$$
$$- (b-d\theta)\big(\mathrm{Var}[\phi(a)] + E[\phi(a)]^2\big)$$

$$= \mathrm{Cov}[\phi(a)a] - (b-d\theta)\mathrm{Var}[\phi(a)]$$
$$+ E[\phi(a)](\mu-(b-d\theta)(E[\phi(a)])). \tag{A3}$$

Inserting (A3) into (A2) and that into (A1), (A1) is now:

$$\hat{\phi} = \arg\max_{\phi} \mathrm{Cov}[\phi(a)a] - (b-d\theta)\mathrm{Var}[\phi(a)]$$
$$+ E[\phi(a)](\mu-(b-d\theta)(E[\phi(a)]))$$
$$- c(\mu-(b-d\theta)E[\phi(a)]) + (E[\phi(a)]-c)d(1-\theta)\hat{p} \tag{A4}$$

$$s.t.\ \mathrm{Cov}[\phi(a)a] - (b-d\theta)\mathrm{Var}[\phi(a)]$$
$$+ E[\phi(a)](\mu-(b-d\theta)(E[\phi(a)]))$$
$$- c(\mu-(b-d\theta)E[\phi(a)])$$
$$+ (E[\phi(a)]-c)d(1-\theta)\hat{p} - f$$
$$\geq (\hat{p}-c)\big(\mu - b\hat{p} + d\big(\theta E[\phi(a)] + (1-\theta)\hat{p}\big)\big)$$
$$\hat{p} = \arg\max_{p}(p-c)\Big(\mu - bp + d\Big(\theta E\Big[\hat{\phi}(a)\Big] + (1-\theta)\hat{p}\Big)\Big).$$

To show that a solution to (A4) is an affine function, let us suppose it is not and then construct an affine function that yields a higher payoff. That contradiction will establish that a non-affine function could not be a solution.

Suppose ϕ^o is a solution to (A4), and it is not affine. Consider an affine function $\tilde{\phi}(a) = \alpha + \gamma a$ with the same mean and variance as ϕ^o : $E\Big[\tilde{\phi}(a)\Big] = E[\phi^o(a)]$ and $\mathrm{Var}[\tilde{\phi}(a)] = \mathrm{Var}[\phi^o(a)]$. To show $\tilde{\phi}$ exists with those properties, I will construct it. First note,

$$E\Big[\tilde{\phi}(a)\Big] = \alpha + \gamma\mu, \mathrm{Var}[\tilde{\phi}(a)] = \gamma^2\sigma^2.$$

We can now solve for (α, γ),

$$\gamma^2\sigma^2 = \mathrm{Var}[\phi^o(a)] \Rightarrow \gamma = \sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}$$
$$\alpha + \gamma\mu = E\big[\phi^o(a)\big] \Rightarrow \alpha = E\big[\phi^o(a)\big] - \left(\sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}\right)\mu.$$

Thus,

$$\tilde{\phi}(a) = E\big[\phi^o(a)\big] - \left(\sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}\right)\mu + \left(\sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}\right)a \tag{A5}$$

is an affine function with the same mean and variance as ϕ^o.

Returning to the problem, we are supposing ϕ^o is a solution to (A2). Given $E\Big[\tilde{\phi}(a)\Big] = E[\phi^o(a)]$ and $\mathrm{Var}[\tilde{\phi}(a)] = \mathrm{Var}[\phi^o(a)]$ then (A2) evaluated at ϕ^o and $\tilde{\phi}$ are identical except for the term $\mathrm{Cov}[\phi(a)a]$. If $\mathrm{Cov}[\tilde{\phi}(a)a] > \mathrm{Cov}[\phi^o(a)a]$ then the objective is higher with $\tilde{\phi}$ and, given the constraint is presumed to be satisfied with ϕ^o, $\mathrm{Cov}[\tilde{\phi}(a)a] > \mathrm{Cov}[\phi^o(a)a]$ implies it is satisfied with $\tilde{\phi}$.

To prove $\mathrm{Cov}[\tilde{\phi}(a)a] > \mathrm{Cov}[\phi^o(a)a]$, let us derive an expression for $\mathrm{Cov}[\tilde{\phi}(a)a]$ using (A5).

$$\mathrm{Cov}[\tilde{\phi}(a)a] = E\Big[\tilde{\phi}(a)a\Big] - E\Big[\tilde{\phi}(a)\Big]\mu$$
$$= E\left[\left(E\big[\phi^o(a)\big] - \left(\sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}\right)\mu + \sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}a\right)a\right]$$
$$- E\left[E\big[\phi^o(a)\big] - \left(\sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}\right)\mu + \sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}a\right]\mu$$
$$= \left(E\big[\phi^o(a)\big] - \left(\sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}\right)\mu\right)\mu$$
$$+ E\left[\sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}a^2\right]$$
$$- \left(E\big[\phi^o(a)\big] - \left(\sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}\right)\mu + \sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}\mu\right)\mu$$

$$= E\big[\phi^o(a)\big]\mu - \left(\sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}\right)\mu^2$$
$$+ \sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}E\big[a^2\big] - E\big[\phi^o(a)\big]\mu$$
$$= E\big[\phi^o(a)\big]\mu - \left(\sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}\right)\mu^2$$
$$+ \sqrt{\frac{\mathrm{Var}[\phi^o(a)]}{\sigma^2}}\big(\sigma^2+\mu^2\big) - E\big[\phi^o(a)\big]\mu,$$

and, therefore,

$$\mathrm{Cov}[\tilde{\phi}(a)a] = \sqrt{\mathrm{Var}[\phi^o(a)]}\sigma.$$

By the Cauchy–Schwarz inequality,

$$\mathrm{Cov}[\phi^o(a)a] \leq \sqrt{\mathrm{Var}[\phi^o(a)]}\sigma$$

and

$$\mathrm{Cov}[\phi^o(a)a] = \sqrt{\mathrm{Var}[\phi^o(a)]}\sigma$$

iff $\phi^o(a)$ and a are linearly dependent which requires that ϕ^o is an affine function.[35] Given ϕ^o is presumed to be a non-affine function then

$$\mathrm{Cov}[\phi^o(a)a] < \sqrt{\mathrm{Var}[\phi^o(a)]}\sigma$$

which implies $\mathrm{Cov}[\phi^o(a)a] < \mathrm{Cov}[\tilde{\phi}(a)a]$.

Even with linear demand, the equilibrium pricing algorithm in Sugaya and Wolitzky [18] is not an affine function of the demand state as it involves pooling at high demand states. This difference in the properties of the pricing algorithm is attributable to the different decisions being modeled. I model a firm's decision to adopt the pricing algorithm, while assuming it implements the algorithm's generated prices. Hence, a firm chooses between setting the algorithm's generated price *for all demand states* and setting a uniform price. Sugaya and Wolitzky [18] model a firm's decision to implement the algorithm's generated prices, while assuming it has adopted the pricing algorithm. Hence, a firm chooses between the algorithm's generated price (and any alternative price) *for each demand state*. Thus, a firm can depart from the algorithm for some but not all demand states. That flexibility affects the design of the pricing algorithm.

Proof of Theorem 1

Theorem 1 is proven through the following steps.

1. Lemma A1 proves a solution to (9–11) has $\gamma = \frac{1}{2(b-d\theta)}$.
2. Given Lemma A1, the problem is redefined as (9–11) with $\gamma = \frac{1}{2(b-d\theta)}$. For the redefined problem, a solution is a value for α which is a fixed point to ψ (which is defined below).
3. Lemma A2 derives some useful properties for the unconstrained problem defined by (9–11) with $\gamma = \frac{1}{2(b-d\theta)}$.
4. Lemma A3 proves there is a fixed point to ψ that lies in $\left(\alpha^I, \alpha^U\right]$.
5. Lemma A4 derives some useful properties of the constraint (10).
6. It is shown that the fixed point derived in step 4 is the unique solution to (9–11) and is as characterized in Theorem 1.

Let us begin with some preliminary results. Solve for a non-adopter's symmetric equilibrium price, p^{NA}, given pricing algorithm $\alpha + \gamma a$.

$$p^{NA} = \arg\max_p \int (p-c)\left(a - bp + d\left(\theta(\alpha+\gamma a) + (1-\theta)p^{NA}\right)\right) \times G'(a)da,$$

which yields

$$p^{NA}(\alpha,\gamma) = \frac{\mu + bc + d\theta(\alpha+\gamma\mu)}{2b - d(1-\theta)}.$$

One can derive a simpler expression for an adopter's profit in place of (9).

$$\begin{aligned}\pi^A(\alpha,\gamma,p) &\equiv \int (\alpha+\gamma a - c)(a - b(\alpha+\gamma a) + d(\theta(\alpha+\gamma a) \\ &\quad +(1-\theta)p))G'(a)da \\ &= (\alpha+\gamma\mu - c)(\mu - (b-d\theta)(\alpha+\gamma\mu) \\ &\quad +d(1-\theta)p) + \gamma(1-(b-d\theta)\gamma)\sigma^2. \end{aligned} \tag{A6}$$

To save on notation, I will use f in place of the fee $\frac{(1+\upsilon)\eta\sigma^2}{4(b-d\theta)}$.

Lemma A1. *A solution to* (9–11) *has* $\hat{\gamma} = \gamma^* \equiv \frac{1}{2(b-d\theta)}$.

Proof. Proof is by contradiction. Suppose, contrary to the lemma, the solution is $\left(\alpha', \gamma'\right)$ where $\gamma' \neq \frac{1}{2(b-d\theta)}$. Consider an alternative pair $\left(\alpha'', \gamma''\right)$ where $\gamma'' = \frac{1}{2(b-d\theta)}$ and $\alpha'' + \gamma''\mu = \alpha' + \gamma'\mu$ so the average price is the same. Note the RHS of the constraint (10) is the same for $\left(\alpha', \gamma'\right)$ and $\left(\alpha'', \gamma''\right)$ as it depends only on the average price, $\alpha + \gamma\mu$. Given the LHS of (10) is the objective in Equation (9) minus f, if the objective is greater with $\left(\alpha'', \gamma''\right)$ than with $\left(\alpha', \gamma'\right)$ then: If the constraint is satisfied with $\left(\alpha', \gamma'\right)$ then it is also satisfied with $\left(\alpha'', \gamma''\right)$. We would then have a contradiction, for it will have been shown there is $(\alpha,\gamma) \neq \left(\alpha', \gamma'\right)$ which yields a higher value for the objective and satisfies the constraint.

Evaluate (A6) at $\left(\alpha', \gamma'\right)$ and $\left(\alpha'', \gamma''\right)$:

$$\begin{aligned}\pi^A\left(\alpha',\gamma',p\right) &= \left(\alpha'+\gamma'\mu - c\right)\left(\mu - (b-d\theta)\left(\alpha'+\gamma'\mu\right)\right. \\ &\quad \left.+\, d(1-\theta)p\right) \\ &\quad + \gamma'\left(1-(b-d\theta)\gamma'\right)\sigma^2, \end{aligned}$$

$$\begin{aligned}\pi^A\left(\alpha'',\gamma'',p\right) &= \left(\alpha''+\gamma''\mu - c\right)\left(\mu - (b-d\theta)\left(\alpha''+\gamma''\mu\right)\right. \\ &\quad \left.+\, d(1-\theta)p\right) + \gamma''\left(1-(b-d\theta)\gamma''\right)\sigma^2. \end{aligned}$$

Since $\alpha'' + \gamma''\mu = \alpha' + \gamma'\mu$ then

$$\begin{aligned}&\pi^A\left(\alpha'',\gamma'',p\right) - \pi^A\left(\alpha',\gamma',p\right) \\ &\quad = \sigma^2\left(\gamma''\left(1-(b-d\theta)\gamma''\right) - \gamma'\left(1-(b-d\theta)\gamma'\right)\right) > 0 \Leftrightarrow \\ &\qquad \gamma''\left(1-(b-d\theta)\gamma''\right) > \gamma'\left(1-(b-d\theta)\gamma'\right). \end{aligned} \tag{A7}$$

Given $\gamma(1-(b-d\theta)\gamma)$ is strictly concave in γ, its maximum is defined by:

$$\frac{\partial\gamma(1-(b-d\theta)\gamma)}{\partial\gamma} = 2d\theta\gamma - 2b\gamma + 1 = 0 \Rightarrow \gamma = \frac{1}{2(b-d\theta)}.$$

Given $\gamma'' = \frac{1}{2(b-d\theta)} \neq \gamma'$, (A7) is satisfied which implies $\pi^A\left(\alpha'',\gamma'',p\right) > \pi^A\left(\alpha',\gamma',p\right)$ and thus contradicts $\left(\alpha',\gamma'\right)$ being a solution to (9–11). □

In light of Lemma A1, a solution to (9–11) is $\gamma = \gamma^*$ and a value for α that is a fixed point to

$$\begin{aligned}\psi(\alpha^o) &\equiv \arg\max_\alpha \pi^A\left(\alpha, \tilde{p}(\alpha^o)\right) \\ &\quad \text{s.t. } \pi^A\left(\alpha,\tilde{p}(\alpha^o)\right) - f \geq \pi^{NA}\left(\alpha,\tilde{p}(\alpha^o)\right), \end{aligned} \tag{A8}$$

where

$$\begin{aligned}\pi^A(\alpha,p) &\equiv (\alpha+\gamma^*\mu - c)\times \\ &\quad (\mu - (b-d\theta)(\alpha+\gamma^*\mu) + d(1-\theta)p) \\ &\quad + \gamma^*(1-(b-d\theta)\gamma^*)\sigma^2 \\ \pi^{NA}(\alpha,p) &\equiv (p-c)(\mu - (b-d(1-\theta))p + d\theta(\alpha+\gamma^*\mu)) \\ \tilde{p}(\alpha^o) &\equiv \frac{\mu + bc + d\theta(\alpha^o + \gamma^*\mu)}{2b - d(1-\theta)}. \end{aligned}$$

Note that (11) has been replaced with the condition $\hat{p} = \tilde{p}(\alpha^o)$.

While the plan is to show ψ has a unique fixed point, it will prove useful to begin with the unconstrained problem, which is a fixed point of $\varphi(\alpha^o)$ where

$$\varphi(\alpha^o) \equiv \arg\max_\alpha \pi^A\left(\alpha, \tilde{p}(\alpha^o)\right).$$

In Lemma A2, recall that α^U is defined in Equation (14).

Lemma A2. $\varphi(\alpha^o) \gtreqless \alpha^o$ *as* $\alpha^o \lesseqgtr \alpha^U$.

Proof. First note $\pi^A\left(\alpha, \tilde{p}(\alpha^o)\right)$ is strictly concave in α:

$$\begin{aligned}&\pi^A\left(\alpha,\tilde{p}(\alpha^o)\right) \\ &\quad = (\alpha+\gamma^*\mu - c)(\mu - (b-d\theta)(\alpha+\gamma^*\mu) \\ &\qquad +d(1-\theta)\tilde{p}(\alpha^o)) \\ &\qquad + \gamma^*(1-(b-d\theta)\gamma^*)\sigma^2 \\ &\quad = \left(\alpha + \left(\frac{1}{2(b-d\theta)}\right)\mu - c\right)(\mu - (b-d\theta) \\ &\qquad \times\left(\alpha + \left(\frac{1}{2(b-d\theta)}\right)\mu\right) + d(1-\theta)\tilde{p}(\alpha^o)\right) \\ &\qquad + \left(\frac{1}{2(b-d\theta)}\right)\left(1-(b-d\theta)\left(\frac{1}{2(b-d\theta)}\right)\right)\sigma^2 \end{aligned}$$

$$\frac{\partial^2\pi^A\left(\alpha,\tilde{p}(\alpha^o)\right)}{\partial\alpha^2} = -2(b-d\theta) < 0.$$

The optimal value of α is then the solution to the first-order condition:

$$\frac{\partial \pi^A\left(\alpha, \widetilde{p}(\alpha^o)\right)}{\partial \alpha} = \mu + (b - d\theta)c - 2(b - d\theta)(\alpha + \gamma^* \mu) + d(1-\theta)\widetilde{p}(\alpha^o) = 0.$$

Solving it for α and substituting the expression for $\widetilde{p}(\alpha^o)$, we have:

$$\varphi(\alpha^o) = \frac{\mu + (b - d\theta)c - 2(b - d\theta)\gamma^* \mu + d(1-\theta)\widetilde{p}(\alpha^o)}{2(b - d\theta)} = \frac{\mu + (b - d\theta)c - 2(b - d\theta)\gamma^* \mu + d(1-\theta)\left(\frac{\mu + bc + d\theta(\alpha^o + \gamma^* \mu)}{2b - d(1-\theta)}\right)}{2(b - d\theta)}.$$

Note that φ is linear and increasing in α^o.

Next let us show $\varphi\left(\alpha^I\right) > \alpha^I$ where recall $\alpha^I = \arg\max_\alpha \pi^A\left(\alpha, \widetilde{p}(\alpha^I)\right) - \pi^{NA}\left(\alpha, \widetilde{p}(\alpha^I)\right)$. First note $\pi^A\left(\alpha, \widetilde{p}(\alpha^o)\right) - \pi^{NA}\left(\alpha, \widetilde{p}(\alpha^o)\right)$ is strictly concave in α:

$$\begin{aligned}
&\pi^A\left(\alpha, \widetilde{p}(\alpha^o)\right) - \pi^{NA}\left(\alpha, \widetilde{p}(\alpha^o)\right) \\
&= (\alpha + \gamma^* \mu - c)(\mu - (b - d\theta)(\alpha + \gamma^* \mu) \\
&\quad + d(1-\theta)\widetilde{p}(\alpha^o)) + \gamma^*(1 - (b - d\theta)\gamma^*)\sigma^2 \\
&\quad - (\widetilde{p}(\alpha^o) - c)(\mu - (b - d(1-\theta)) \cdot \widetilde{p}(\alpha^o) + d\theta(\alpha + \gamma^* \mu)),
\end{aligned}$$

$$\frac{\partial^2\left(\pi^A\left(\alpha, \widetilde{p}(\alpha^o)\right) - \pi^{NA}\left(\alpha, \widetilde{p}(\alpha^o)\right)\right)}{\partial \alpha^2} = -2(b - d\theta) < 0.$$

Given strict concavity and that α^I maximizes $\pi^A\left(\alpha, \widetilde{p}(\alpha^o)\right) - \pi^{NA}\left(\alpha, \widetilde{p}(\alpha^o)\right)$, it follows:

$$\frac{\partial\left(\pi^A\left(\alpha^I, \widetilde{p}(\alpha^I)\right) - \pi^{NA}\left(\alpha^I, \widetilde{p}(\alpha^I)\right)\right)}{\partial \alpha} = 0. \tag{A9}$$

Since

$$\frac{\partial \pi^{NA}\left(\alpha^I, \widetilde{p}(\alpha^o)\right)}{\partial \alpha} = d\theta\left(\widetilde{p}(\alpha^o) - c\right) > 0,$$

then (A9) implies

$$\frac{\partial \pi^A\left(\alpha^I, \widetilde{p}(\alpha^o)\right)}{\partial \alpha} > 0.$$

We then have:

$$\frac{\partial \pi^A\left(\alpha^I, \widetilde{p}(\alpha^I)\right)}{\partial \alpha} > 0 = \frac{\partial \pi^A\left(\varphi\left(\alpha^I\right), \widetilde{p}(\alpha^I)\right)}{\partial \alpha}.$$

Given $\pi^A\left(\alpha, \widetilde{p}(\alpha^o)\left(\alpha^I\right)\right)$ is strictly concave in α then $\varphi\left(\alpha^I\right) > \alpha^I$.

In Harrington [19], $\alpha^o = \varphi(\alpha^o)$ is solved and there it is shown it has a unique solution which is α^U as defined in Equation (14). Note that $\alpha^I < \alpha^U$ since, as expressed in Equation (14), α^U equals α^I plus a positive term.

Given $\varphi\left(\alpha^I\right) > \alpha^I, \varphi\left(\alpha^U\right) = \alpha^U$, and $\varphi(\alpha^o)$ is linear in α^o, we have: $\varphi(\alpha^o) \gtreqqless \alpha^o$ as $\alpha^o \lesseqqgtr \alpha^U$. □

Lemma A3 proves the existence of a solution to (A8).

Lemma A3. *There exists $\alpha^o \in \left(\alpha^I, \alpha^U\right]$ such that $\alpha^o = \psi(\alpha^o)$.*

Proof. The proof strategy is to show: (1) $\psi(\alpha^o)$ is a continuous function of α^o; (2) $\psi(\alpha^o) \leq \alpha^U \forall \alpha^o \in \left[\alpha^I, \alpha^U\right]$; and (3) $\alpha^I < \psi\left(\alpha^I\right)$. By the continuity of ψ, $\alpha^I < \psi\left(\alpha^I\right)$ and $\psi\left(\alpha^U\right) \leq \alpha^U$ prove the theorem.

To show $\psi(\alpha^o)$ is a continuous function of α^o, first note the objective $\pi^A\left(\alpha, \widetilde{p}(\alpha^o)\right)$ is continuous in α and α^o and has already been shown to be strictly concave in α. Turning to the constraint

$$\omega(\alpha, \alpha^o) = \pi^A\left(\alpha, \gamma, \widetilde{p}(\alpha^o)\right) - f - \pi^{NA}\left(\alpha, \gamma, \widetilde{p}(\alpha^o)\right) \geq 0,$$

$\omega(\alpha, \alpha^o)$ is continuous in α and α^o and convex-valued because, as has been shown, $\pi^A\left(\alpha, \gamma, \widetilde{p}(\alpha^o)\right) - \pi^{NA}\left(\alpha, \gamma, \widetilde{p}(\alpha^o)\right)$ is strictly concave in α. Consequently, the optimum $\psi(\alpha^o)$ is uniquely defined and continuous in α^o.

Let us prove $\psi(\alpha^o) \leq \alpha^U \forall \alpha^o \in \left[\alpha^I, \alpha^U\right]$ by showing $\psi(\alpha^o)$ is bounded above by $\varphi(\alpha^o)$ and, from Lemma A3, $\varphi(\alpha^o) \leq \alpha^U \forall \alpha^o \in \left[\alpha^I, \alpha^U\right]$. If the constraint is satisfied at $\alpha = \varphi(\alpha^o)$ then $\psi(\alpha^o) = \varphi(\alpha^o)$ and, therefore, $\psi(\alpha^o) \leq \alpha^U$. Next, suppose the constraint is not satisfied at $\alpha = \varphi(\alpha^o)$ and let us evaluate $\pi^A\left(\alpha, \widetilde{p}(\alpha^o)\right) - \pi^{NA}\left(\alpha, \widetilde{p}(\alpha^o)\right)$ at $\alpha = \varphi(\alpha^o)$. Note that

$$\frac{\partial \pi^A\left(\varphi(\alpha^o), \widetilde{p}(\alpha^o)\right)}{\partial \alpha} = 0,$$

and

$$\frac{\partial \pi^{NA}\left(\varphi(\alpha^o), \widetilde{p}(\alpha^o)\right)}{\partial \alpha} = d\theta\left(\widetilde{p}(\alpha^o) - c\right) > 0.$$

Hence,

$$\frac{\partial\left(\pi^A\left(\varphi(\alpha^o), \widetilde{p}(\alpha^o)\right) - \pi^{NA}\left(\varphi(\alpha^o), \widetilde{p}(\alpha^o)\right)\right)}{\partial \alpha} < 0. \tag{A10}$$

As it is presumed the constraint is violated at $\alpha = \varphi(\alpha^o)$,

$$\pi^A\left(\varphi(\alpha^o), \widetilde{p}(\alpha^o)\right) - \pi^{NA}\left(\varphi(\alpha^o), \widetilde{p}(\alpha^o)\right) < f, \tag{A11}$$

Equations (A10) and (A11) and the strict concavity of $\pi^A\left(\alpha, \widetilde{p}(\alpha^o)\right) - \pi^{NA}\left(\alpha, \widetilde{p}(\alpha^o)\right)$ in α imply $\psi(\alpha^o) < \varphi(\alpha^o)$ in order for the constraint to be satisfied at $\alpha = \psi(\alpha^o)$. It has then been shown $\psi(\alpha^o) \leq \alpha^U \forall \alpha^o \in \left[\alpha^I, \alpha^U\right]$.

Finally, let us show $\alpha^I < \psi\left(\alpha^I\right)$. As

$$\alpha^I = \arg\max_\alpha \pi^A\left(\alpha, \widetilde{p}(\alpha^I)\right) - \pi^{NA}\left(\alpha, \widetilde{p}(\alpha^I)\right)$$

and it is assumed

$$\pi^A\left(\alpha^I, \widetilde{p}(\alpha^I)\right) - \pi^{NA}\left(\alpha^I, \widetilde{p}(\alpha^I)\right) > f,$$

then the constraint is strictly satisfied at $\alpha = \alpha^I$ when $\alpha^o = \alpha^I$. Given $\alpha^I < \varphi\left(\alpha^I\right)$ and the strict concavity of $\pi^A\left(\alpha, \widetilde{p}(\alpha^I)\right)$ in α, it follows $\partial \pi^A\left(\alpha^I, \widetilde{p}(\alpha^I)\right)/\partial \alpha > 0$. As the objective $\pi^A\left(\alpha^I, \widetilde{p}(\alpha^I)\right)$ is increasing in α and the constraint is strictly satisfied at $(\alpha, \alpha^o) = \left(\alpha^I, \alpha^I\right)$, we have $\psi\left(\alpha^I\right) > \alpha^I$. □

To prove the uniqueness of this solution, we will need some properties on the constraint in Equation (A8). Here, it is defined at a solution so $\alpha = \alpha^o$:

$$\omega(\alpha) \equiv \pi^A\left(\alpha, \widetilde{p}(\alpha)\right) - f - \pi^{NA}\left(\alpha, \widetilde{p}(\alpha)\right) \geq 0.$$

In Lemma A4, recall that $\widetilde{\alpha}$ is defined in Equation (13).

Lemma A4. *For $\alpha \geq \alpha^I$, $\omega(\alpha) \gtreqqless 0$ as $\alpha \lesseqqgtr \widetilde{\alpha}$.*

Proof. One can show:

$$\begin{aligned}
&\omega(\alpha) \\
&= \left(\frac{1}{4(b - d\theta)}\right)(\mu + 2(b - d\theta)(\alpha - c)) \times \\
&\quad \left(\mu - 2(b - d\theta)\alpha + 2d(1-\theta)\left(\frac{\mu + bc + d\theta\left(\alpha + \frac{\mu}{2(b - d\theta)}\right)}{2b - d(1-\theta)}\right)\right) \\
&\quad + \frac{\sigma^2}{4(b - d\theta)} - \frac{b(\mu - (b - d(1-\theta))c + d\theta\left(\alpha + \frac{\mu}{2(b - d\theta)}\right))^2}{(2b - d(1-\theta))^2} - f,
\end{aligned}$$

and that $\omega'(\alpha) < 0$ if and only if $\alpha > \alpha^I$. Also note:

$$\omega''(\alpha) = -\frac{2b(2b-d)^2}{(2b-d(1-\theta))^2} < 0.$$

As noted in Section 2.2,

$$\pi^A\left(\alpha^I, \widetilde{p}(\alpha^I)\right) - \pi^{NA}\left(\alpha^I, \widetilde{p}(\alpha^I)\right) = \frac{\sigma^2}{4(b-d\theta)}.$$

Hence,

$$\begin{aligned}\omega\left(\alpha^I\right) > 0 \Leftrightarrow\ & \pi^A\left(\alpha^I, \widetilde{p}(\alpha^I)\right) \\ & - \pi^{NA}\left(\alpha^I, \widetilde{p}(\alpha^I)\right) - f > 0 \\ \Leftrightarrow\ & \frac{\sigma^2}{4(b-d\theta)} > f,\end{aligned}$$

which is true by assumption. Given $\omega\left(\alpha^I\right) > 0, \omega'(\alpha) < 0$, and $\omega''(\alpha) < 0$, there exists a unique $\widetilde{\alpha} > \alpha^I$ such that $\omega(\alpha) \lesseqgtr 0$ as $\alpha \gtreqless \widetilde{\alpha}$. It is tedious but reasonably straightforward to show the expression for $\widetilde{\alpha}$ in Equation (13) is the solution to $\omega(\alpha) = 0$.[36] □

To complete the proof of Theorem 1, we need to show that the fixed point is unique and is characterized in the statement of the theorem. Recall a solution to (9–11) is a value for α that is a fixed point to $\psi(\alpha^o)$, where

$$\begin{aligned}\psi(\alpha^o) \equiv \arg\max_{\alpha}\ & \pi^A\left(\alpha, \widetilde{p}(\alpha^o)\right)\ s.t. \pi^A\left(\alpha, \widetilde{p}(\alpha^o)\right) \\ & - f - \pi^{NA}\left(\alpha, \widetilde{p}(\alpha^o)\right) \geq 0,\end{aligned}$$

and $\gamma = \frac{1}{2(b-d\theta)}$.

Let us first prove there is no fixed point greater than α^U by showing $\psi(\alpha^o) < \alpha^o \forall \alpha^o > \alpha^U$. Suppose the latter is not true, so $\exists \alpha' > \alpha^U$ such that $\psi\left(\alpha'\right) \geq \alpha'$. By Lemma A2, $\alpha' > \alpha^U$ implies $\varphi\left(\alpha'\right) < \alpha'$ which means $\varphi\left(\alpha'\right) < \psi\left(\alpha'\right)$ and therefore

$$\frac{\partial \pi^A\left(\psi\left(\alpha'\right), \gamma, \widetilde{p}(\alpha')\right)}{\partial \alpha} < 0, \tag{A12}$$

by the strict concavity of $\pi^A\left(\alpha, \gamma, \widetilde{p}(\alpha')\right)$ in α. Along with

$$\frac{\partial \pi^{NA}\left(\psi\left(\alpha'\right), \gamma, \widetilde{p}(\alpha')\right)}{\partial \alpha} = (\widetilde{p}(\alpha^o) - c)d\theta > 0,$$

we have

$$\begin{aligned}\frac{\partial \omega\left(\psi\left(\alpha'\right), \alpha'\right)}{\partial \alpha} = & \frac{\partial \pi^A\left(\psi\left(\alpha'\right), \gamma, \widetilde{p}(\alpha')\right)}{\partial \alpha} \\ & - \frac{\partial \pi^{NA}\left(\psi\left(\alpha'\right), \gamma, \widetilde{p}(\alpha')\right)}{\partial \alpha} < 0.\end{aligned}$$

Thus, $\omega\left(\psi\left(\alpha'\right), \alpha'\right) \geq 0$ implies $\omega\left(\psi\left(\alpha'\right) - \varepsilon, \alpha'\right) > 0$ for $\varepsilon > 0$ and small. Given (A12), the objective is higher, and the constraint is satisfied at $\alpha = \psi\left(\alpha'\right) - \varepsilon$ which contradicts $\psi\left(\alpha'\right)$ being a solution. I conclude there is no fixed point greater than α^U.

Next, let us show there is no fixed point less than α^I. Suppose, contrary to the claim, $\alpha' = \psi\left(\alpha'\right)$ and $\alpha' < \alpha^I$. Given $\omega\left(\alpha^I\right) > 0$, $\alpha' < \alpha^I$ and Lemma A3 imply $\omega\left(\alpha'\right) > 0$. As the constraint is not binding, then $\psi\left(\alpha'\right) = \varphi\left(\alpha'\right)$. But, by Lemma A3, $\alpha' < \alpha^U$ implies $\varphi\left(\alpha'\right) > \alpha'$ which means $\psi\left(\alpha'\right) \neq \alpha'$ which contradicts the supposition that α' is a fixed point. I conclude there is no fixed point less than α^I.

We now need only concern ourselves with fixed points in $\left[\alpha^I, \alpha^U\right]$.

Suppose α' is a solution to (9–11) and $\alpha' < \alpha^U$. Let us show the constraint must bind: $\omega\left(\alpha'\right) = 0$. If it does not bind then it implies $\alpha' = \varphi\left(\alpha'\right)$. But then $\alpha' = \alpha^U$ which is a contradiction. Hence, if α' is a solution and $\alpha' < \alpha^U$ then $\omega\left(\alpha'\right) = 0$. Furthermore, by Lemma A4, it is the unique solution less than α^U. In addition, using Lemma A4, α^U is not a solution because $\omega\left(\alpha^U\right) < 0$ follows from $\omega\left(\alpha'\right) = 0$ and $\alpha' < \alpha^U$. We conclude that if α' is a solution and $\alpha' < \alpha^U$ then $\alpha' = \widetilde{\alpha}$ and is the unique solution.

Now suppose α' is a solution and $\alpha' = \alpha^U$. By the preceding argument, if there is a solution less than α^U then it is the unique solution which would contradict α^U being a solution.

Given Lemma A4, if $\widetilde{\alpha} < \alpha^U$ then $\omega\left(\alpha^U\right) < 0$ and the unique solution is $\widetilde{\alpha}$; and if $\alpha^U \leq \widetilde{\alpha}$ then $\omega\left(\alpha^U\right) \geq 0$ and the unique solution is α^U. This proves $\widehat{\alpha} = \min\left\{\widetilde{\alpha}, \alpha^U\right\}$.

Proofs of Theorem 2–5

Proof of Theorem 2. By Theorem 1, we know $\widehat{\alpha} = \min\left\{\widetilde{\alpha}, \alpha^U\right\}$. Using their expressions, $\widetilde{\alpha} > \alpha^U$ takes the form:

$$\begin{aligned}& \frac{2b(b-d\theta)c + d\mu(1-2\theta)}{2(2b-d)(b-d\theta)} \\ & \quad + \left(\frac{(2b-d(1-\theta))\sqrt{1-(1+\upsilon)\eta}}{2(2b-d)\sqrt{b^2-bd\theta}}\right)\sigma \\ & > \frac{\begin{array}{c}(b-d\theta)c\left(2b(2b-d(1+\theta)) + d^2\theta(1-\theta)\right) \\ +d(1-\theta)(2b(\mu+bc) - d\theta(\mu+(b+d\theta)c))\end{array}}{2(b-d\theta)\left(2b(2b-d(1+\theta)) + d^2\theta(1-\theta)\right)}.\end{aligned}$$

After some manipulations, we find it is equivalent to

$$\sigma^2 > \left(\frac{1}{1-(1+\upsilon)\eta}\right)\left(\frac{2d\theta(\mu-(b-d)c)\sqrt{b(b-d\theta)}}{2b(2b-d(1+\theta)) + d^2\theta(1-\theta)}\right)^2 > 0,$$

which proves the theorem. □

Proof of Theorem 3. The solution is unconstrained if and only if

$$\begin{aligned}& (\widetilde{\alpha} =)\frac{2b(b-d\theta)c + d\mu(1-2\theta)}{2(2b-d)(b-d\theta)} \\ & \quad + \left(\frac{(2b-d(1-\theta))\sqrt{1-(1+\upsilon)\eta}}{2(2b-d)\sqrt{b^2-bd\theta}}\right)\sigma \\ & > \frac{\begin{array}{c}(b-d\theta)c\left(2b(2b-d(1+\theta)) + d^2\theta(1-\theta)\right) \\ +d(1-\theta)(2b(\mu+bc) - d\theta(\mu+(b+d\theta)c))\end{array}}{2(b-d\theta)\left(2b(2b-d(1+\theta)) + d^2\theta(1-\theta)\right)}(= \alpha^U).\end{aligned}$$

As $\theta \to 0$, the condition for an unconstrained solution is:

$$\lim_{\theta\to 0}\widetilde{\alpha} = \frac{2b^2c + d\mu}{2b(2b-d)} + \frac{\sigma\sqrt{1-(1+\upsilon)\eta}}{2b} > \frac{2b^2c + d\mu}{2b(2b-d)} = \lim_{\theta\to 0}\alpha^U,$$

which is true. Hence, $\exists \varepsilon > 0$ such that $\widehat{\alpha} = \alpha^U \forall \theta \in [0, \varepsilon]$.

As $\theta \to 1$, the condition for a constrained solution is:

$$\begin{aligned}\lim_{\theta\to 1}\alpha^U & = \lim_{\theta\to 1}\frac{\begin{array}{c}(b-d\theta)c\left(2b(2b-d(1+\theta)) + d^2\theta(1-\theta)\right) \\ +d(1-\theta)(2b(\mu+bc) - d\theta(\mu+(b+d\theta)c))\end{array}}{2(b-d\theta)\left(2b(2b-d(1+\theta)) + d^2\theta(1-\theta)\right)} \\ & > \lim_{\theta\to 1}\frac{2b(b-d\theta)c + d\mu(1-2\theta)}{2(2b-d)(b-d\theta)} \\ & \quad + \left(\frac{(2b-d(1-\theta))\sqrt{1-(1+\upsilon)\eta}}{2(2b-d)\sqrt{b^2-bd\theta}}\right)\sigma \\ & = \lim_{\theta\to 1}\widetilde{\alpha},\end{aligned}$$

which can be shown to be equivalent to

$$\sigma^2 < \frac{d^2(\mu - (b-d)c)^2}{4b(b-d)(1-(1+\upsilon)\eta)}. \tag{A13}$$

Hence, if (A13) holds then $\exists \varepsilon > 0$ such that $\hat{\alpha} = \tilde{\alpha} \forall \theta \in [1-\varepsilon, 1]$. □

Proof of Theorem 4. Since $\tilde{\alpha}$ and α^U are both continuous in θ then $\min\left\{\tilde{\alpha} + \frac{\mu}{2(b-d\theta)}, \alpha^U + \frac{\mu}{2(b-d\theta)}\right\}$ is continuous in θ. If, in addition, $\tilde{\alpha} + \frac{\mu}{2(b-d\theta)}$ and $\alpha^U + \frac{\mu}{2(b-d\theta)}$ are both increasing in θ then

$$\min\left\{\tilde{\alpha} + \frac{\mu}{2(b-d\theta)}, \alpha^U + \frac{\mu}{2(b-d\theta)}\right\}$$

is increasing in θ from which it follows the average collusive price and supracompetitive markup are increasing in θ.

Since

$$\begin{aligned}
&\tilde{\alpha} + \frac{\mu}{2(b-d\theta)} \\
&\quad= \frac{2b(b-d\theta)c + d\mu(1-2\theta)}{2(2b-d)(b-d\theta)} \\
&\qquad+ \left(\frac{(2b-d(1-\theta))\sqrt{1-(1+\upsilon)\eta}}{2(2b-d)\sqrt{b^2-bd\theta}}\right)\sigma \\
&\qquad+ \frac{\mu}{2(b-d\theta)} \\
&\quad= \frac{\mu+bc}{2b-d} + \left(\frac{(2b-d(1-\theta))\sqrt{1-(1+\upsilon)\eta}}{2(2b-d)\sqrt{b^2-bd\theta}}\right)\sigma,
\end{aligned}$$

then

$$\frac{\partial\left(\tilde{\alpha} + \frac{\mu}{2(b-d\theta)}\right)}{\partial\theta} = \frac{bd(4b-d(1+\theta))\sqrt{1-(1+\upsilon)\eta}}{4(b(b-d\theta))^{3/2}(2b-d)} > 0.$$

Next consider:

$$\begin{aligned}
&\alpha^U + \frac{\mu}{2(b-d\theta)} \\
&\quad= \frac{\mu+bc}{2b-d} + \frac{d\theta(2b-d(1-\theta))(\mu-(b-d)c)}{(2b-d)(2b(2b-d(1+\theta)) + d^2\theta(1-\theta))},
\end{aligned}$$

from which it follows

$$\frac{\partial\left(\alpha^U + \frac{\mu}{2(b-d\theta)}\right)}{\partial\theta} = \frac{2bd(2b-d(1-2\theta))(\mu-(b-d)c)}{(2b(2b-d(1+\theta)) + d^2\theta(1-\theta))^2} > 0.$$

□

Proof of Theorem 5. First, note that the collusive pricing algorithm, when the participation constraint binds, is

$$\begin{aligned}
\tilde{\alpha} + \left(\frac{1}{2(b-d\theta)}\right)a = \hat{\alpha} &+ \left(\frac{(2b-d(1-\theta))\sqrt{1-(1+\upsilon)\eta}}{2(2b-d)\sqrt{b^2-bd\theta}}\right)\sigma \\
&+ \left(\frac{1}{2(b-d\theta)}\right)a.
\end{aligned}$$

It is immediate that the average collusive price and supracompetitive markup are increasing in σ^2 and decreasing in η and υ. Since $\tilde{\alpha}$ and α^U are both continuous in σ^2 then $\min\left\{\tilde{\alpha} + \frac{\mu}{2(b-d\theta)}, \alpha^U + \frac{\mu}{2(b-d\theta)}\right\}$ is continuous in σ^2, η, and υ. Given α^U does not depend on σ^2, η, and υ, the result is proven. □