Appendix

■ Proofs of Theorems 1–4 and Lemma 1 follow.

Proof of Theorem 1. Suppose Γ is empty. As the choice set is the singleton $\{No\ Cartel\}$, the OSSPE price path is $\hat{P}$ forever. For the remainder of the proof, suppose Γ is nonempty. Consider the payoff function in (2). Since $\pi(\cdot)$ and $x(\cdot)$ are bounded functions and $\delta, \beta \in (0, 1)$, the payoff function is defined for all price paths. The payoff function is continuous in $\{P^t\}_{t=1}^{\infty}$ by the continuity of $\pi(\cdot)$, $x(\cdot)$, and $\phi(\cdot)$. To show that Γ is a compact set, first note that it is a subset of Ω^{∞}, which, by the compactness of Ω and Tychonoff's Product Theorem, is itself compact. The left-hand-side expression of the ICC is continuous in $\{P^t\}_{t=1}^{\infty}$. Under (i) of Assumption A1, the right-hand-side expression is continuous (using Assumption A5). Under (ii), the right-hand side takes the form

$$\max_{P_i \leq P^t} n\pi(P_i) + \delta\phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right)\left[\frac{\widehat{\pi}}{1-\delta} - \sum_{j=1}^{t-1}\beta^{t-j}\gamma x(P^j) - F\right]$$
$$+ \delta\left[1 - \phi\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), P^{t-1}\right)\right] V_i^{mpe}\left(\left(P^t, \ldots, P_i, \ldots, P^t\right), \sum_{j=1}^{t-1}\beta^{t-j}\gamma x(P^j)\right),$$

which is also continuous in P^t. It follows that Γ is a closed set. Since Γ is a closed subset of a compact set, Γ is compact. There is then a solution to (2) as it involves maximizing a continuous function over a nonempty compact set. If the associated payoff exceeds $\hat{\pi}/(1-\delta)$, then such a solution is an OSSPE price path. If it does not exceed $\hat{\pi}/(1-\delta)$, then an OSSPE price path is $\hat{P}$ forever. *Q.E.D.*

Proof of Theorem 2. The proof is made up of two steps. Suppose $\{\overline{P}^t\}_{t=1}^{\infty}$ is an OSSPE path. First, it is shown that if $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'}$, then it is IC to keep price constant and thereby price at $\overline{P}^{t'}$ in $t'+1$. Note that the ICC when price is raised to $\overline{P}^{t'}$ and when it is kept constant at $\overline{P}^{t'}$ is identical in terms of current profit and the future payoff but differs only in terms of the current probability of detection. With Assumption B1, cheating on the cartel more favorably affects the probability of detection when price is raised to $\overline{P}^{t'}$ than when it is kept fixed at $\overline{P}^{t'}$. Thus, if it is IC to raise price to some level, then it is IC to keep it at that level. Second, if, contrary to the theorem, this price path has a decreasing subsequence, then, by the first step, one can substitute that decreasing subsequence with a constant price path that is IC and yields a strictly higher payoff. This produces the desired contradiction.

Given this OSSPE, let $V(\overline{P}^t)$ denote the associated payoff starting with period $t+1$.[1] In performing the first step, let us initially show that if $\overline{P}^{t'-1} \leq \overline{P}^{t'}$ and $V(\overline{P}^{t'-1}) \geq \hat{\pi}/(1-\delta)$, then it is IC to keep price at $\overline{P}^{t'}$. There are two cases to consider: (i) $V(\overline{P}^{t'-1}) > V(\overline{P}^{t'})$, and (ii) $V(\overline{P}^{t'-1}) \leq V(\overline{P}^{t'})$. Starting with case (i) and recognizing that the left-hand side of (A1) is $V(\overline{P}^{t'-1})$, we have

$$\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\frac{\widehat{\pi}}{1-\delta} - F\right] + \delta[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)]V\left(\overline{P}^{t'}\right) > V\left(\overline{P}^{t'}\right). \tag{A1}$$

This implies

$$\frac{\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)[(\hat{\pi}/(1-\delta)) - F]}{1 - \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right]} > V\left(\overline{P}^{t'}\right). \tag{A2}$$

Substituting the left-hand side of (A2) for $V(\overline{P}^{t'})$ in the expression for $V(\overline{P}^{t'-1})$ on the left-hand side of (A1), the following upper bound for $V(\overline{P}^{t'-1})$ is derived:

$$V\left(\overline{P}^{t'-1}\right) < \pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\frac{\hat{\pi}}{1-\delta} - F\right]$$
$$+ \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right]\left[\frac{\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\dfrac{\hat{\pi}}{1-\delta} - F\right]}{1 - \delta\left(1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right)}\right].$$

[1] Throughout this article, $V(\cdot)$ denotes the payoff in period t from an OSSPE. This is not a value function and it is required to be defined only for values of the state variables on the OSSPE path.

Rearranging yields

$$V\left(\overline{P}^{t'-1}\right) < \frac{\pi\left(\overline{P}^{t'}\right) + \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\frac{\hat{\pi}}{1-\delta} - F\right]}{1 - \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right]}, \tag{A3}$$

which gives us an upper bound on $V(\overline{P}^{t'-1})$.

Now consider a constant price path of $\overline{P}^{t'}$ starting in period $t'+1$. The payoff, denoted $W(\overline{P}^{t'})$, is defined by

$$W\left(\overline{P}^{t'}\right) = \pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\left[\frac{\hat{\pi}}{1-\delta} - F\right] + \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\right] W\left(\overline{P}^{t'}\right),$$

and, solving for $W(\overline{P}^{t'})$,

$$W\left(\overline{P}^{t'}\right) = \frac{\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\left[\frac{\hat{\pi}}{1-\delta} - F\right]}{1 - \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\right]}. \tag{A4}$$

$W(\overline{P}^{t'}) > V(\overline{P}^{t'-1})$ follows from (A3) and (A4), since $\phi(\overline{P}^{t'}, \overline{P}^{t'}) \leq \phi(\overline{P}^{t'}, \overline{P}^{t'-1})$ by Assumption A4. Given that, by supposition, $V(\overline{P}^{t'-1}) \geq \hat{\pi}/(1-\delta)$, then $W(\overline{P}^{t'}) \geq \hat{\pi}/(1-\delta)$. The next step is to show that this constant price path is IC. The ICC for period t' for the original OSSPE path is

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\frac{\widehat{\pi}}{1-\delta} - F\right] + \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right] V\left(\overline{P}^{t'}\right) \\
&\quad \geq \max_{P_i \in \Omega} \pi\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[\frac{\hat{\pi}}{1-\delta} - F\right] \\
&\qquad + \delta\left[1 - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right).
\end{aligned} \tag{A5}$$

As $W(\overline{P}^{t'}) > V(\overline{P}^{t'-1}) > V(\overline{P}^{t'})$, then (A5) continues to hold if $W(\overline{P}^{t'})$ replaces $V(\overline{P}^{t'})$:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\frac{\widehat{\pi}}{1-\delta} - F\right] + \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right] W\left(\overline{P}^{t'}\right) \\
&\quad \geq \max_{P_i \in \Omega} \pi\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[\frac{\hat{\pi}}{1-\delta} - F\right] \\
&\qquad + \delta\left[1 - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right).
\end{aligned} \tag{A6}$$

Now consider the ICC for a constant price path of $\overline{P}^{t'}$:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\left[\frac{\widehat{\pi}}{1-\delta} - F\right] + \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'}\right)\right] W\left(\overline{P}^{t'}\right) \\
&\quad \geq \max_{P_i \in \Omega} \pi\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'}\right)\left[\frac{\hat{\pi}}{1-\delta} - F\right] \\
&\qquad + \delta\left[1 - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'}\right)\right] V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right).
\end{aligned} \tag{A7}$$

I want to show that (A6) implies (A7). Note that we need only be concerned with $P_i < \overline{P}^{t'}$, as deviating with a price in excess of $\overline{P}^{t'}$ cannot yield a higher payoff than colluding, as current profit is weakly lower, the probability of detection is weakly higher, and, by Assumption B2, $W(\overline{P}^{t'}) \geq \hat{\pi}/(1-\delta)$ implies $W(\overline{P}^{t'}) \geq V_i^{mpe}(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'})$ so that the MPE payoff is weakly lower than the future collusive payoff. Rearranging (A6) and (A7), I want to show that

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right) - \pi\left(P_i, \overline{P}^{t'}\right) + \delta\left[W\left(\overline{P}^{t'}\right) - V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right)\right] \\
&\quad \geq \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left\{W\left(\overline{P}^{t'}\right) - \left[\frac{\hat{\pi}}{1-\delta} - F\right]\right\} - \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right) \\
&\qquad \times \left\{V_i^{mpe}\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right) - \left[\frac{\hat{\pi}}{1-\delta} - F\right]\right\}
\end{aligned} \tag{A8}$$

implies

$$\begin{aligned}&\pi\left(\overline{P}^{t'}\right)-\overline{\pi}\left(P_i,\overline{P}^{t'}\right)+\delta\left[W\left(\overline{P}^{t'}\right)-V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)\right]\\&\quad\geq\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\left\{W\left(\overline{P}^{t'}\right)-\left[\frac{\hat{\pi}}{1-\delta}-F\right]\right\}-\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\\&\quad\times\left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\frac{\hat{\pi}}{1-\delta}-F\right]\right\},\qquad\forall\,P_i<\overline{P}^{t'}.\end{aligned}\tag{A9}$$

As the left-hand sides of (A8) and (A9) are identical, (A8) implies (A9) if the right-hand side of (A8) is at least as great as the right-hand side of (A9). Rearranging that latter inequality yields

$$\begin{aligned}&\left[\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]\left\{W\left(\overline{P}^{t'}\right)-\left[\frac{\hat{\pi}}{1-\delta}-F\right]\right\}\\&\quad\geq\left[\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]\\&\quad\times\left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-\left[\frac{\hat{\pi}}{1-\delta}-F\right]\right\}.\end{aligned}\tag{A10}$$

Since $W(\overline{P}^{t'})\geq\hat{\pi}/(1-\delta)\geq V_i^{mpe}(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'})\geq(\hat{\pi}/(1-\delta))-F$ and $\phi(\overline{P}^{t'},\overline{P}^{t'-1})\geq\phi(\overline{P}^{t'},\overline{P}^{t'})$, then (A10) holds if

$$\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\geq\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right),$$

or, equivalently,

$$\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)\geq\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right).\tag{A11}$$

Given that $\overline{P}^{t'}\geq P_i,\overline{P}^{t'-1}$, (A11) holds by Assumption B1. Having shown that a constant price path of $\overline{P}^{t'}$ starting from $t'+1$ is IC and yields a payoff strictly greater than $V(\overline{P}^{t'})$, we have a contradiction that the original price path is an OSSPE path. Therefore, if $\overline{P}^{t'}\geq\overline{P}^{t'-1}$ and $V(\overline{P}^{t'-1})\geq\hat{\pi}/(1-\delta)$, it cannot be true that $V(\overline{P}^{t'-1})>V(\overline{P}^{t'})$.

Let us now examine case (ii): $V(\overline{P}^{t'-1})\leq V(\overline{P}^{t'})$. Consider keeping price at $\overline{P}^{t'}$ in period $t'+1$ but then continuing with the original OSSPE path. The ICC at $t'+1$ is

$$\begin{aligned}&\pi\left(\overline{P}^{t'}\right)+\delta\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\left[\frac{\widehat{\pi}}{1-\delta}-F\right]+\delta\left[1-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]V\left(\overline{P}^{t'}\right)\\&\quad\geq\max_{P_i\in\Omega}\overline{\pi}\left(P_i,\overline{P}^{t'}\right)+\delta\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\left[\frac{\hat{\pi}}{1-\delta}-F\right]\\&\quad+\delta\left[1-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right).\end{aligned}\tag{A12}$$

Using the same series of steps as with case (i), (A5) implies (A12) if

$$\begin{aligned}&\left[\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)-\phi\left(\overline{P}^{t'},\overline{P}^{t'}\right)\right]\left\{V\left(\overline{P}^{t'}\right)-\left[\frac{\hat{\pi}}{1-\delta}-F\right]\right\}\\&\quad\geq\left[\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]\\&\quad\times\left\{V_i^{mpe}\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right)-[(\hat{\pi}/(1-\delta))-F]\right\}.\end{aligned}$$

The same argument is used to show that this inequality holds. The important point to note is that $V(\overline{P}^{t'})\geq V_i^{mpe}(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'})$ because $V(\overline{P}^{t'})\geq V(\overline{P}^{t'-1})\geq\hat{\pi}/(1-\delta)$. Hence, if $\overline{P}^{t'}\geq\overline{P}^{t'-1}$, then it is IC to keep price at $\overline{P}^{t'}$ and, in addition, $V(\overline{P}^{t'})\geq\widehat{\pi}/(1-\delta)$.

To summarize, it has been shown that on an OSSPE path, if $V(\overline{P}^{t'-1}) \geq \hat{\pi}/(1-\delta)$ and $\overline{P}^{t'-1} \leq \overline{P}^{t'}$, then (i) it is IC to keep price at $\overline{P}^{t'}$, and (ii) $V(\overline{P}^{t'}) \geq \hat{\pi}/(1-\delta)$. Arguing by strong induction, I will show that if the price path is nondecreasing, then it is IC to keep price constant in the future. First note that by the conditions of an OSSPE, $V(P^0) \geq \hat{\pi}/(1-\delta)$. Since, by supposition, $P^0 \leq \overline{P}^1$, it then follows that it is IC to keep price at $\overline{P}^1$. Also note that $V(\overline{P}^1) \geq \hat{\pi}/(1-\delta)$. Now suppose $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'}$. By strong induction, it follows from $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'-1}$ that $V(\overline{P}^{t'-1}) \geq \hat{\pi}/(1-\delta)$. Since $V(\overline{P}^{t'-1}) \geq \hat{\pi}/(1-\delta)$ and, by supposition, $\overline{P}^{t'} \geq \overline{P}^{t'-1}$, it is IC to keep price at $\overline{P}^{t'}$. This shows that on a nondecreasing price path, it is IC to keep price constant. Also note that as long as an OSSPE price path is nondecreasing, then so is the value to colluding: if $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'}$, then $V(P^0) \leq \cdots \leq V(\overline{P}^{t'})$.

Armed with this property, the second step is to suppose that $\{\overline{P}^t\}_{t=1}^{\infty}$ is not nondecreasing and show that there exists another IC path that yields a strictly higher payoff. Suppose the price path declines at some time and let $t'+1$ be the first period in which it does so, $P^0 \leq \overline{P}^1 \leq \cdots \leq \overline{P}^{t'} > \overline{P}^{t'+1}$. Define $t''+1$ as the first period after t' for which price is at least as great as in t': $\overline{P}^t < \overline{P}^{t'}$ $\forall\, t \in \{t'+1, \ldots, t''\}$ and $\overline{P}^{t''+1} \geq \overline{P}^{t'}$. t'' might be ∞. Now consider an alternative price path in which price equals $\overline{P}^{t'}$ for periods $t'+1, \ldots, t''$ and is identical to the original path starting at $t''+1$. First note that this alternative path yields a strictly higher payoff than the original path, since it generates strictly higher profit in periods $t'+1, \ldots, t''$ (here I use the property that price does not exceed P^m so that a higher price means higher profit) and the same profit thereafter. Furthermore, it results in a weakly lower probability of detection in periods $t'+1, \ldots, t''+1$ because, with this alternative path, price doesn't change over $t'+1, \ldots, t''$ and, with respect to $t''+1$, the price rise is $\overline{P}^{t''+1} - \overline{P}^{t'}$ with the alternative path as opposed to a higher price rise of $\overline{P}^{t''+1} - \overline{P}^{t''}$ with the original path, which means a weakly lower probability of detection.

Having established that this alternative price path yields a strictly higher payoff, let me argue that it is IC. Consider incentive compatibility over $t'+1, \ldots, t''$. If $t'=0$, then, since $P^0 = \hat{P}$, a constant price path of $P^{t'}$ over $t'+1, \ldots, t''$ is certainly IC. If $t' \geq 1$, then $\overline{P}^{t'-1} \leq \overline{P}^{t'}$ and, by our previous analysis, a constant price of $\overline{P}^{t'}$ starting with period $t'+1$ is IC. It is also IC for periods after $t''+1$, since the previous period's price and the current period's price are the same as with the original path, which, by supposition, is IC. The only remaining ICC is for period $t''+1$. The period $t''+1$ price is the same for both paths, but with the original path the lagged price is $\overline{P}^{t''}$ and with the alternative path it is $\overline{P}^{t'}$, where $\overline{P}^{t'} > \overline{P}^{t''}$. The ICC for $t''+1$ for the original path can be rearranged to

$$\begin{aligned}
&\pi\left(\overline{P}^{t''+1}\right) - \overline{\pi}\left(P_i, \overline{P}^{t''+1}\right) + \delta\left[V\left(\overline{P}^{t''+1}\right) - V_i^{mpe}\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right)\right] \\
&\quad \geq \delta\phi\left(\overline{P}^{t''+1}, \overline{P}^{t''}\right)\left\{V\left(\overline{P}^{t''+1}\right) - \left[\frac{\hat{\pi}}{1-\delta} + F\right]\right\} - \delta\phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t''}\right) \\
&\quad\quad \times \left\{V_i^{mpe}\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right) - \left[\frac{\hat{\pi}}{1-\delta} - F\right]\right\}, \qquad \forall\, P_i \leq \overline{P}^{t''+1}.
\end{aligned} \tag{A13}$$

The ICC for the alternative path at $t''+1$ can be rearranged to

$$\begin{aligned}
&\pi\left(\overline{P}^{t''+1}\right) - \overline{\pi}\left(P_i, \overline{P}^{t''+1}\right) + \delta\left[V\left(\overline{P}^{t''+1}\right) - V_i^{mpe}\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right)\right] \\
&\quad \geq \delta\phi\left(\overline{P}^{t''+1}, \overline{P}^{t'}\right)\left\{V\left(\overline{P}^{t''+1}\right) - \left[\frac{\hat{\pi}}{1-\delta)} - F\right]\right\} - \delta\phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t'}\right) \\
&\quad\quad \times \left\{V_i^{mpe}\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right) - \left[\frac{\hat{\pi}}{1-\delta} - F\right]\right\}, \qquad \forall\, P_i \leq \overline{P}^{t''+1}.
\end{aligned} \tag{A14}$$

I then want to show that the right-hand side of (A13) is at least as great as the right-hand side of (A14):

$$\begin{aligned}
&\left[\phi\left(\overline{P}^{t''+1}, \overline{P}^{t''}\right) - \phi\left(\overline{P}^{t''+1}, \overline{P}^{t'}\right)\right]\left\{V\left(\overline{P}^{t''+1}\right) - \left[\frac{\hat{\pi}}{1-\delta} - F\right]\right\} \\
&\quad \geq \left[\phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t''}\right) - \phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t'}\right)\right] \\
&\quad\quad \times \left\{V_i^{mpe}\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right) - \left[\frac{\hat{\pi}}{1-\delta} - F\right]\right\}, \qquad \forall\, P_i \leq \overline{P}^{t''+1}.
\end{aligned} \tag{A15}$$

Let me first argue that $V(\overline{P}^{t''+1}) \geq V_i^{mpe}(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1})$. As the OSSPE price path is nondecreasing over $1, \ldots, t'$, then, by our earlier argument, $V(\overline{P}^{t'}) \geq \widehat{\pi}/(1-\delta)$. Next note that since a constant price path of $\overline{P}^{t'}$ is IC—and recalling that $W(\overline{P}^{t'})$ denotes the associated payoff—then the conditions of an OSSPE imply $V(\overline{P}^{t'}) \geq W(\overline{P}^{t'})$. Since the expected income stream from the OSSPE path is less than that from the constant price path over $t'+1, \ldots, t''$ (recall that the former generates strictly lower profit and a weakly higher probability of detection in those periods), it must deliver a higher payoff stream after t''. Since $V(\overline{P}^{t''})$ is the payoff associated with the stream after t'', it follows that $V(\overline{P}^{t''}) > V(\overline{P}^{t'})$. We then have $V(\overline{P}^{t''}) \geq \hat{\pi}/(1-\delta)$, and since $\overline{P}^{t''+1} \geq \overline{P}^{t''}$ implies $V(\overline{P}^{t''+1}) \geq \widehat{\pi}/(1-\delta)$, it follows that $V(\overline{P}^{t''+1}) \geq V_i^{mpe}(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1})$. Since $\phi(\overline{P}^{t''+1}, \overline{P}^{t''}) \geq \phi(\overline{P}^{t''+1}, \overline{P}^{t'})$, a sufficient condition for (A15) to hold is

$$\phi\left(\overline{P}^{t''+1}, \overline{P}^{t''}\right) - \phi\left(\overline{P}^{t''+1}, \overline{P}^{t'}\right) \geq \phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t''}\right) - \phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t'}\right),$$

or, equivalently,

$$\phi\left(\overline{P}^{t''+1}, \overline{P}^{t''}\right) - \phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t''}\right) \geq \phi\left(\overline{P}^{t''+1}, \overline{P}^{t'}\right) - \phi\left(\left(\overline{P}^{t''+1}, \ldots, P_i, \ldots, \overline{P}^{t''+1}\right), \overline{P}^{t'}\right).$$

Since $\overline{P}^{t''+1} > \overline{P}^{t'} > \overline{P}^{t''}$ and $\overline{P}^{t''+1} > P_i$, this condition follows from Assumption B1. *Q.E.D.*

Proof of Lemma 1. The method of proof is to presume that $\exists t'$ such that $\overline{X}^{t'-1} < \overline{X}^{t'} > \overline{X}^{t'+1}$ and derive a contradiction. Associated with such a path of damages is a relatively high level of current damages (and, therefore, a high price) in t', $\overline{X}^{t'} - \beta\overline{X}^{t'-1}$, and a relatively low level, $\overline{X}^{t'+1} - \beta\overline{X}^{t'}$, in $t'+1$. However, as $\pi(\xi(\cdot))$ is concave in damages, then it is more profitable to have more incremental changes in damages. More specifically, it is shown that if current damages of $\overline{X}^{t'} - \beta\overline{X}^{t'-1}$ is preferred to $\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}$ in t', then it must be true that $\overline{X}^{t'} - \beta\overline{X}^{t'}$ is preferred to $\overline{X}^{t'+1} - \beta\overline{X}^{t'}$ in $t'+1$, which gives us a contradiction.

A critical property that will be used is that if, on an OSSPE path, the cartel prices at P' and the damage state variable at the end of the period is X', then pricing at P with end-of-period damages of X is also IC if $P \leq P'$ and $X \leq X'$. To see this, consider the ICC for $(P^t, X^t) = (P', X')$:

$$\pi(P') + \sum_{\tau=t+1}^{\infty} \delta^{\tau-t}(1-\phi^0)^{\tau-t}\left[\pi(P^t) - \theta^c \gamma x(P^\tau)\right] + \kappa^c\left[\frac{\hat{\pi}}{1-\delta} - F\right] - \theta^c \beta X'$$
$$\geq \overline{\pi}\left(\psi(P'), P'\right) + \delta\left(\hat{\pi}/(1-\delta)\right) - \theta^d X' + \theta^d \gamma x(P') - \kappa^d F.$$

Since, by deviating rather than colluding, a firm avoids current damages of $\gamma x(P')$, if the end-of-period damages are X' when a firm colludes, then they are $[X' - \gamma x(P')]$ when it deviates. By Assumptions C1–C2, the left-hand side decreases at a weakly faster rate with respect to X' than the right-hand side. Hence, if X' is replaced with a lower value for the damage variable, this condition still holds. By Assumptions C3–C4, $\overline{\pi}(\psi(P), P) + \theta^d \gamma x(P) - \pi(P)$ is increasing in P. Hence, this ICC holds if P' is replaced with a lower price. I conclude that, on an OSSPE path, if (P^t, X^t) is replaced with a lower price and/or lower damage variable, then the ICC at t still holds.

Since $X^0 = 0$, if $\overline{X}^1 = 0$, then, by the stationarity of the policy function, $\overline{X}^t = 0 \forall t$ and thus, trivially, damages are nondecreasing.[2] Next suppose that $X^0 < \overline{X}^1$. If Lemma 1 is not true, then $\exists t' \geq 1$ such that $X^0 < \overline{X}^1 < \cdots < \overline{X}^{t'} > \overline{X}^{t'+1}$. (Note that if damages are constant from one period to the next, then they are constant in all future periods by stationarity.) Given the path of damages on an OSSPE path, the associated prices in t' and $t'+1$ are defined by $\overline{P}^{t'} = \xi(\overline{X}^{t'} - \beta\overline{X}^{t'-1})$ and $\overline{P}^{t'+1} = \xi(\overline{X}^{t'+1} - \beta\overline{X}^{t'})$. That is, $\xi(\overline{X}^{t'} - \beta\overline{X}^{t'-1})$ is the price that results in damages of $\overline{X}^{t'}$ given inherited damages of $\beta\overline{X}^{t'-1}$.

[2] The assumption $X^0 = 0$ could be replaced with the condition that, on the optimal path, $X^0 < \overline{X}^1$.

Since, by supposition, $\overline{X}^{t'} > \overline{X}^{t'+1}$ and furthermore $\overline{X}^{t'+1} \geq \beta\overline{X}^{t'} > \beta\overline{X}^{t'-1}$, then $\overline{X}^{t'+1} \in [\beta\overline{X}^{t'-1}, \overline{X}^{t'}]$. Hence, it was feasible to set price in t' so that damages equalled $\overline{X}^{t'+1}$ at t' and the price that would have done this is $\xi(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1})$. Since $\overline{X}^{t'} - \beta\overline{X}^{t'-1} > \overline{X}^{t'+1} - \beta\overline{X}^{t'-1}$ and ξ is increasing, then $\xi(\overline{X}^{t'} - \beta\overline{X}^{t'-1}) > \xi(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1})$. Given that, by supposition, charging a price of $\xi(\overline{X}^{t'} - \beta\overline{X}^{t'-1})$ with resulting total damages of $\overline{X}^{t'}$ is IC (as it is part of an OSSPE), then the price-damage pair $(\xi(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}), \overline{X}^{t'+1})$ is also IC, as it involves a lower collusive price and lower damages. Since $(\xi(\overline{X}^{t'} - \beta\overline{X}^{t'-1}), \overline{X}^{t'})$ was selected in t' and, as just argued, the cartel could have chosen $(\xi(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}), \overline{X}^{t'+1})$, I conclude that the former yields at least as high a payoff. Letting $V(X)$ denote the payoff associated with the OSSPE when damages are X, the previous statement is then represented as

$$\begin{aligned}
&\pi\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right)\right) + \delta\phi^0\left[\frac{\hat{\pi}}{1-\delta} - \overline{X}^{t'} - F\right] + \delta(1-\phi^0)V\left(\overline{X}^{t'}\right)\\
&\quad \geq \pi\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right)\right) + \delta\phi^0\left[\frac{\hat{\pi}}{1-\delta} - \overline{X}^{t'+1} - F\right] + \delta(1-\phi^0)V\left(\overline{X}^{t'+1}\right)\\
\Leftrightarrow\quad&\\
&\pi\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right)\right) - \pi\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right)\right)\\
&\quad \geq \delta(1-\phi^0)\left[V\left(\overline{X}^{t'+1}\right) - V\left(\overline{X}^{t'}\right)\right] + \delta\phi^0\left(\overline{X}^{t'} - \overline{X}^{t'+1}\right).
\end{aligned} \tag{A16}$$

Next note that $(\xi(\overline{X}^{t'} - \beta\overline{X}^{t'-1}), \overline{X}^{t'})$ being IC implies $(\xi(\overline{X}^{t'} - \beta\overline{X}^{t'}), \overline{X}^{t'})$ is as well, since $\xi(\overline{X}^{t'} - \beta\overline{X}^{t'-1}) > \xi(\overline{X}^{t'} - \beta\overline{X}^{t'})$. Given that $(\xi(\overline{X}^{t'+1} - \beta\overline{X}^{t'}), \overline{X}^{t'+1})$ was chosen in $t'+1$, it follows that $(\xi(\overline{X}^{t'+1} - \beta\overline{X}^{t'}), \overline{X}^{t'+1})$ yields at least as high a payoff as $(\xi(\overline{X}^{t'} - \beta\overline{X}^{t'}), \overline{X}^{t'})$:

$$\begin{aligned}
&\pi\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'}\right)\right) + \delta\phi^0\left[(\hat{\pi}/(1-\delta)) - \overline{X}^{t'+1} - F\right] + \delta(1-\phi^0)V\left(\overline{X}^{t'+1}\right)\\
&\quad \geq \pi\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'}\right)\right) + \delta\phi^0\left[(\hat{\pi}/(1-\delta)) - \overline{X}^{t'} - F\right] + \delta(1-\phi^0)V\left(\overline{X}^{t'}\right)\\
\Leftrightarrow\quad&\\
&\delta(1-\phi^0)\left[V\left(\overline{X}^{t'+1}\right) - V\left(\overline{X}^{t'}\right)\right] + \delta\phi^0\left(\overline{X}^{t'} - \overline{X}^{t'+1}\right)\\
&\quad \geq \pi\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'}\right)\right) - \pi\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'}\right)\right).
\end{aligned} \tag{A17}$$

(A16)–(A17) imply

$$\pi\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'-1}\right)\right) - \pi\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}\right)\right) \geq \pi\left(\xi\left(\overline{X}^{t'} - \beta\overline{X}^{t'}\right)\right) - \pi\left(\xi\left(\overline{X}^{t'+1} - \beta\overline{X}^{t'}\right)\right). \tag{A18}$$

Note that the difference in the arguments on the left-hand side of (A18) is $(\overline{X}^{t'} - \beta\overline{X}^{t'-1}) - (\overline{X}^{t'+1} - \beta\overline{X}^{t'-1}) = \overline{X}^{t'} - \overline{X}^{t'+1}$ and on the right-hand side is $(\overline{X}^{t'} - \beta\overline{X}^{t'}) - (\overline{X}^{t'+1} - \beta\overline{X}^{t'}) = \overline{X}^{t'} - \overline{X}^{t'+1}$. By the concavity of $\pi(\xi(\cdot))$, it then follows from (A18) that $\overline{X}^{t'+1} - \beta\overline{X}^{t'-1} \leq \overline{X}^{t'+1} - \beta\overline{X}^{t'} \Leftrightarrow \overline{X}^{t'} \leq \overline{X}^{t'-1}$, which is a contradiction. This proves that $\overline{X}^{t}$ is nondecreasing on an OSSPE path. *Q.E.D.*

Proof of Theorem 3. Let $\{\overline{X}^{t}\}_{t=1}^{\infty}$ denote the path of damages associated with $\{\overline{P}^{t}\}_{t=1}^{\infty}$. Recall that $(P^0, X^0) = (\hat{P}, 0)$.[3] If $\overline{P}^1 \leq \hat{P}$, then, since $X^0 = 0$, $\overline{X}^1 = 0$ (by Assumption C4). Hence, by stationarity, an OSSPE price path then involves pricing at $\overline{P}^1$ in period 2 and every period thereafter. As this contradicts the optimality of colluding, it is inferred that $\overline{P}^1 > \hat{P}$ and, therefore, $\overline{P}^1 > P^0$.[4]

If Theorem 3 is not true, then $\exists\, t' \geq 1$ such that $P^0 < \overline{P}^1 \geq \cdots \geq \overline{P}^{t'} < \overline{P}^{t'+1}$. For the OSSPE price path under consideration, let $\overline{P}^{t'} = P'$ and $\overline{P}^{t'+1} = P''$, where $P' < P''$. The analysis will involve comparing the original price path—$\{\overline{P}^1, \ldots, \overline{P}^{t'-1}, P', P'', \overline{P}^{t'+2}, \ldots\}$—with an alternative price path—$\{\overline{P}^1, \ldots, \overline{P}^{t'-1}, P'', P', \overline{P}^{t'+2}, \ldots\}$—which has the prices in t' and $t'+1$ switched. It will be shown that if a price path has price rise from one period to the next, then an alternative price path in which those two prices are switched yields a strictly higher collusive payoff, and if the original price path was IC, then so is this one. This contradicts the original price path being induced by an OSSPE and thus contradicts the supposition that an OSSPE price path has an increasing subsequence after period 1.

[3] The assumption $P^0 = \hat{P}$ can be replaced with $\overline{P}^1 > P^0$ on the optimal path.

[4] If $F > 0$, then colluding and pricing at or below $\hat{P}$ is clearly inferior to not colluding. If $F = 0$, then it could be optimal to collude and price at $\hat{P}$, though that is a nongeneric result.

The first step is to show that an OSSPE price path is bounded from above by P^+ (which is defined in Assumption C6). Suppose not, so that in some period price exceeds P^+. Consider an alternative price path that is identical except that it has a price of P^+ in those periods for which price exceeded P^+. By Assumption C6, the collusive payoff, which is expressed in (3), is strictly higher, since $\pi(P^+) - \theta^c \gamma x(P^+)$ exceeds the comparable expression when price exceeded P^+. By Assumption C4, accumulated damages are lower. As ICCs are loosened when damages are reduced, if the original price path is IC, then so is this one. In that a price path has been constructed that generates a higher payoff and is IC, it contradicts the supposition that the original path was generated by an OSSPE. I conclude that an OSSPE price path is bounded from above by P^+.

Given $P' < P'' \leq P^+$, it follows from Assumption C6 that $\pi(P'') - \theta^c \gamma x(P'') > \pi(P') - \theta^c \gamma x(P')$. Inspection of (3) then reveals that, due to discounting, the alternative price path yields a strictly higher payoff, as it has the cartel receive $\pi(P'') - \theta^c \gamma x(P'')$ in period t' and $\pi(P') - \theta^c \gamma x(P')$ in $t' + 1$, which is the reverse of the original path. The remainder of the proof involves showing that if the original price path is IC, then so is the alternative price path.

I begin with the supposition that the original path is IC in all periods. With the alternative path, the ICCs over periods $1, \ldots, t' - 1$ are still satisfied, since the collusive payoff is higher and the deviation payoff is unchanged. Next consider the period-t constraint where $t \geq t' + 2$. As the current and future price path is the same as with the original path, the only difference in the constraint is lagged damages. Note that accumulated damages at t, where $t \geq t' + 2$, under the alternative path and under the original path are identical in all terms except for the damages incurred in periods t' and $t' + 1$. The difference between the accumulated damages at t, where $t \geq t' + 2$, under the alternative path and under the original path then equals

$$\left[\beta^{t-t'} \gamma x(P'') + \beta^{t-t'-1} \gamma x(P')\right] - \left[\beta^{t-t'} \gamma x(P') + \beta^{t-t'-1} \gamma x(P'')\right] = -\beta^{t-t'-1}(1-\beta)\gamma\left[x(P'') - x(P')\right] < 0.$$

Since, compared to the original path, the alternative path substitutes higher current damages in t' for lower ones in $t' + 1$, accumulated damages are lower after $t' + 1$. Given that damages are lower under the alternative price path, the path is IC for $t \geq t' + 2$.

Next consider the ICC at $t' + 1$. With the original price path, price is P'' and damages are $\beta^2 \overline{X}^{t'-1} + \beta \gamma x(P') + \gamma x(P'')$ at $t' + 1$. With the alternative price path, price is P' and damages are $\beta^2 \overline{X}^{t'-1} + \beta \gamma x(P'') + \gamma x(P')$. As price is lower then, by Assumption C3, this loosens the ICC. As damages are lower, this also serves to loosen the ICC. I conclude that the ICC is satisfied at $t' + 1$ for the alternative price path.

Finally, consider the ICC at t'. Using Lemma 1, it will be shown that if the original price path is IC at $t' + 1$, then the alternative path is IC at t'. As an initial step, compare the damages at $t' + 1$ for the original path with those at t' for the alternative path. The latter is weakly smaller if and only if $\beta \overline{X}^{t'} + \gamma x(P'') \geq \beta \overline{X}^{t'-1} + \gamma x(P'')$. As $\overline{X}^{t'-1} \leq \overline{X}^{t'}$ by Lemma 1, this is then indeed true. Since then damages at t' for the alternative path are weakly lower than damages at $t' + 1$ for the original path, *ceteris paribus*, if the original path is IC at $t' + 1$, then the alternative path is IC at t'. For the next step, recall that the collusive payoff at t' for the alternative path exceeds the collusive payoff at t' for the original path. Since $\overline{X}^{t'} \leq \overline{X}^{t'+1}$, it must then be true, for the original path, that $V(\overline{X}^{t'}) \geq V(\overline{X}^{t'+1})$.[5] Holding fixed the level of accumulated damages, it follows that the collusive payoff at t' for the alternative path exceeds the collusive payoff at $t' + 1$ for the original path. Still holding fixed the level of accumulated damages, since the price at t' for the alternative path is the same as the price at $t' + 1$ for the original path, the deviation payoffs are the same. Finally, since the accumulated damages at t' for the alternative path are weakly lower than the accumulated damages at $t' + 1$ for the original path, the ICC being satisfied at $t' + 1$ for the original path then implies it holds at t' for the alternative path.

It has then been shown that the incentive compatibility of the original price path implies the incentive compatibility of the alternative price path. As the latter yields a strictly higher payoff, this contradicts the original path being generated by an OSSPE and thereby establishes that an OSSPE price path cannot have an increasing subsequence after period 1. *Q.E.D.*

Proof of Theorem 4. Most of the proof works to show that if $\{\overline{P}^t\}_{t=1}^{\infty}$ is an OSSPE price path, then it converges. Define $\mathcal{P}^t \equiv \max\{\overline{P}^0, \overline{P}^1, \ldots, \overline{P}^t\}$ to be the maximum price over the first t periods. As an initial step, it is shown that on an OSSPE path, if the current period's price is at least as great as all past prices, $\overline{P}^{t'} \geq \mathcal{P}^{t'-1}$, then $\pi(\overline{P}^{t'})/(1-\delta)$ is a lower bound on the value in that period: $V(\overline{P}^{t'}, \overline{X}^{t'}) \geq \pi(\overline{P}^{t'})/(1-\delta)$, where $\overline{X}^t$ is the value of the state variable on the OSSPE path. Intuitively, if it was IC to change price to $\overline{P}^{t'}$, then it is IC to keep price at $\overline{P}^{t'}$, since the probability of detection is zero from doing so (by Assumption D2). The next step argues that, generally, $\pi(\mathcal{P}^t)/(1-\delta)$ is a lower bound on the equilibrium payoff. Since $\{\mathcal{P}^t\}_{t=1}^{\infty}$ and $\{\pi(\mathcal{P}^t)/(1-\delta)\}_{t=1}^{\infty}$ are both nondecreasing bounded sequences (with the latter following from the former because the price space has an upper bound of P^m), they have a limit. From this we can argue that $\{\overline{P}^t\}_{t=1}^{\infty}$ has a limit. It is then straightforward to show that $\lim_{t\to\infty} \overline{P}^t = P^*$.

[5] The reason is that at t', the cartel can use the price path starting at $t' + 1$ and, since damages are weakly lower in t', the collusive payoff must be weakly higher.

Assume $\overline{P}^{t'} \geq \mathcal{P}^{t'-1}$, in which case $\overline{P}^{t'} \geq \overline{P}^{t'-1}$. The ICC for period t' is

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\frac{\widehat{\pi}}{1-\delta} - \beta\overline{X}^{t'-1} - \gamma x\left(\overline{P}^{t'}\right) - F\right] \\
&\quad + \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right] V\left(\overline{P}^{t'}, \beta\overline{X}^{t'-1} + \gamma x\left(\overline{P}^{t'}\right)\right) \\
&\quad \geq \max_{P_i \in \Omega} \pi\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[\frac{\hat{\pi}}{1-\delta} - \beta\overline{X}^{t'-1} - F\right] \\
&\quad + \delta\left[1 - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \beta\overline{X}^{t'-1}\right).
\end{aligned} \tag{A19}$$

We want to make two substitutions in (A19). First, replace $[(\hat{\pi}/(1-\delta)) - \beta\overline{X}^{t'-1} - \gamma x(\overline{P}^{t'}) - F]$ on the left-hand side with $[(\hat{\pi}/(1-\delta)) - \beta\overline{X}^{t'-1} - F]$. Second, suppose, contrary to the claim that $\pi(\overline{P}^{t'})/(1-\delta)$ is a lower bound on the collusive payoff, we have $V(\overline{P}^{t'}, \beta\overline{X}^{t'-1} + \gamma x(\overline{P}^{t'})) < \pi(\overline{P}^{t'})/(1-\delta)$ and replace $V(\overline{P}^{t'}, \beta\overline{X}^{t'-1} + \gamma x(\overline{P}^{t'}))$ with $\pi(\overline{P}^{t'})/(1-\delta)$ on the left-hand side of (A19). If (A19) holds, then it is still true after these two substitutions:

$$\begin{aligned}
&\pi\left(\overline{P}^{t'}\right) + \delta\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\left[\frac{\widehat{\pi}}{1-\delta} - \beta\overline{X}^{t'-1} - F\right] \\
&\quad + \delta\left[1 - \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right] \pi\left(\overline{P}^{t'}\right)/(1-\delta) \\
&\quad \geq \max_{P_i \in \Omega} \pi\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[\frac{\hat{\pi}}{1-\delta} - \beta\overline{X}^{t'-1} - F\right] \\
&\quad + \delta\left[1 - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \beta\overline{X}^{t'-1}\right).
\end{aligned} \tag{A20}$$

The objective is to show that pricing at $\overline{P}^{t'}$ from $t'+1$ onward is IC and thus $\pi(\overline{P}^{t'})/(1-\delta)$ is a lower bound on $V\left(\overline{P}^{t'}, \overline{X}^{t'}\right)$, which gives us the desired contradiction.

As an alternative price path, consider the firm maintaining price at the t' level, that is, pricing at $\overline{P}^{t'}$ in period $t, \forall t \geq t'+1$. The ICC for period $t'+1$ is

$$\begin{aligned}
\pi\left(\overline{P}^{t'}\right) + \delta\frac{\pi\left(\overline{P}^{t'}\right)}{1-\delta} &\geq \max_{P_i \in \Omega} \pi\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[\frac{\hat{\pi}}{1-\delta} - \beta\overline{X}^{t'} - F\right] \\
&\quad + \delta\left[1 - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \beta\overline{X}^{t'}\right),
\end{aligned} \tag{A21}$$

where $\overline{X}^{t'} = \beta\overline{X}^{t'-1} + \gamma x(\overline{P}^{t'})$. Note that $\overline{X}^{t'} \geq \overline{X}^{t'-1}$ since $\overline{P}^{t'}$ is the highest price charged thus far. In addition, damages are no longer present in the collusive payoff as, by Assumption D2, $\phi(\overline{P}^{t'}, \overline{P}^{t'}) = 0$. For both (A20) and (A21), the ICC holds when $P_i > \overline{P}^{t'}$, as pricing above $\overline{P}^{t'}$ weakly lowers current profit (by Assumption A1) and weakly raises the probability of detection (by Assumption A4), and it will be shown that the MPE payoff does not exceed the collusive payoff.

I want to show that (A20) implies (A21), which will establish that if the original price path was IC at t', then so is a price of $\overline{P}^{t'}$ at $t'+1$. Since the right-hand side of (A21) is nonincreasing in damages (using Assumption D4), a sufficient condition for (A21) to hold is

$$\begin{aligned}
\pi\left(\overline{P}^{t'}\right) + \delta\frac{\pi\left(\overline{P}^{t'}\right)}{1-\delta} &\geq \max_{P_i \in \Omega} \pi\left(P_i, \overline{P}^{t'}\right) + \delta\phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\left[\frac{\hat{\pi}}{1-\delta} - \beta\overline{X}^{t'-1} - F\right] \\
&\quad + \delta\left[1 - \phi\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \overline{P}^{t'-1}\right)\right] V_i^{mpe}\left(\left(\overline{P}^{t'}, \ldots, P_i, \ldots, \overline{P}^{t'}\right), \beta\overline{X}^{t'-1}\right),
\end{aligned} \tag{A22}$$

where $\beta\overline{X}^{t'}$ has been replaced with $\beta\overline{X}^{t'-1}$. Let us then show that (A20) implies (A22). This is true if the right-hand side minus the left-hand side of (A22) is at least as great as the right-hand side minus the left-hand side of (A20), which, after some manipulation, is

$$\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\left\{\frac{\pi\left(\overline{P}^{t'}\right)}{1-\delta}-\left[\frac{\hat{\pi}}{1-\delta}-\beta\overline{X}^{t'-1}-F\right]\right\}$$
$$\geq\left[\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right)\right]$$
$$\times\left\{V_i^{mpe}\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\beta\overline{X}^{t'-1}\right)-\left[(\hat{\pi}/(1-\delta))-\beta\overline{X}^{t'-1}-F\right]\right\}. \tag{A23}$$

As Assumption D4 implies

$$\frac{\pi\left(\overline{P}^{t'}\right)}{1-\delta}\geq V_i^{mpe}\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\beta\overline{X}^{t'-1}\right)\geq\frac{\hat{\pi}}{1-\delta}-\beta\overline{X}^{t'-1}-F,$$

then (A23) holds if

$$\phi\left(\overline{P}^{t'},\overline{P}^{t'-1}\right)\geq\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'-1}\right)-\phi\left(\left(\overline{P}^{t'},\ldots,P_i,\ldots,\overline{P}^{t'}\right),\overline{P}^{t'}\right). \tag{A24}$$

Since $\overline{P}^{t'} > P_i$ and $\overline{P}^{t'} \geq \overline{P}^{t'-1}$, then (A24) is true by Assumption D3. We can conclude that a constant price path of $\overline{P}^{t'}$ is IC in period $t'+1$. As far as $t > t'+1$, the ICC is as specified in (A21) except that $\overline{X}^{t'}$ is replaced with a weakly higher level of damages. Since the right-hand side of (A21) is decreasing in damages and the left-hand side is independent of them, the ICC holds. In summary, if pricing at $\overline{P}^{t'}$ is IC in t', where $\overline{P}^{t'}$ exceeds all past prices, then a constant price path of $\overline{P}^{t'}$ starting in period $t'+1$ is IC. This implies $V(\overline{P}^{t'},\overline{X}^{t'})\geq\pi(\overline{P}^{t'})/(1-\delta)$, which gives us our desired contradiction. We have then showed that if $\overline{P}^{t'}\geq\mathcal{P}^{t'-1}$, then $V(\overline{P}^{t'},\overline{X}^{t'})\geq\pi(\overline{P}^{t'})/(1-\delta)$.

The next step is to show that, for all periods, a lower bound on the value function at the end of period t is $\pi(\mathcal{P}^t)/(1-\delta)$. The proof is by induction. Start with period t' and suppose that a lower bound on the value function is $\pi(\mathcal{P}^{t'})/(1-\delta)$. Note that t' exists since

$$V\left(P^0,X^0\right)\geq\frac{\pi(P^0)}{1-\delta}=\frac{\pi(\mathcal{P}^0)}{1-\delta}.$$

If $\overline{P}^{t'+1}\geq\mathcal{P}^{t'}$, then the result is immediate by the previous analysis. Next suppose $\overline{P}^{t'+1}<\mathcal{P}^{t'}$. By definition,

$$V\left(\overline{P}^{t'},\overline{X}^{t'}\right)=\pi\left(\overline{P}^{t'+1}\right)+\delta\phi\left(\overline{P}^{t'+1},\overline{P}^{t'}\right)\left[\frac{\hat{\pi}}{1-\delta}-\overline{X}^{t'+1}-F\right]+\delta\left[1-\phi\left(\overline{P}^{t'+1},\overline{P}^{t'}\right)\right]V\left(\overline{P}^{t'+1},\overline{X}^{t'+1}\right).$$

Since, by the inductive step, $V(\overline{P}^{t'},\overline{X}^{t'})\geq\pi(\mathcal{P}^{t'})/(1-\delta)$, then

$$\pi\left(\overline{P}^{t'+1}\right)+\delta\phi\left(\overline{P}^{t'+1},\overline{P}^{t'}\right)\left[\frac{\hat{\pi}}{1-\delta}-\overline{X}^{t'+1}-F\right]+\delta\left[1-\phi\left(\overline{P}^{t'+1},\overline{P}^{t'}\right)\right]V\left(\overline{P}^{t'+1},\overline{X}^{t'+1}\right)\geq\frac{\pi\left(\mathcal{P}^{t'}\right)}{1-\delta}. \tag{A25}$$

Given $\overline{P}^{t'+1}<\mathcal{P}^{t'}$, then $\pi(\overline{P}^{t'+1})<\pi(\mathcal{P}^{t'})$ (here I use the fact that the upper bound on the price space is P^m), which, using (A25), implies

$$\delta\phi\left(\overline{P}^{t'+1},\overline{P}^{t'}\right)\left[\frac{\hat{\pi}}{1-\delta}-\overline{X}^{t'+1}-F\right]+\delta\left[1-\phi\left(\overline{P}^{t'+1},\overline{P}^{t'}\right)\right]V\left(\overline{P}^{t'+1},\overline{X}^{t'+1}\right)>\delta\frac{\pi\left(\mathcal{P}^{t'}\right)}{1-\delta}. \tag{A26}$$

Given that

$$V\left(\overline{P}^{t'+1},\overline{X}^{t'+1}\right)\geq\frac{\hat{\pi}}{1-\delta}-\overline{X}^{t'+1}-F,$$

then (A26) implies

$$V\left(\overline{P}^{t'+1}, \overline{X}^{t'+1}\right) > \frac{\pi\left(\mathcal{P}^{t'}\right)}{1-\delta}.$$

Since $\mathcal{P}^{t'+1} = \mathcal{P}^{t'}$ when $\overline{P}^{t'+1} < \mathcal{P}^{t'}$, we then have

$$V\left(\overline{P}^{t'+1}, \overline{X}^{t'+1}\right) > \frac{\pi\left(\mathcal{P}^{t'+1}\right)}{1-\delta},$$

which is the desired result.

For an OSSPE path, $\pi\left(\mathcal{P}^t\right)/(1-\delta)$ is then a lower bound for $V(\overline{P}^t, \overline{X}^t)$. Since π is increasing in price (here I use the fact that the price path does not exceed P^m) and $\mathcal{P}^t$ is nondecreasing over time (being the maximum of all prices over the first t periods), this lower bound for the value function is a nondecreasing sequence. As it has an upper bound of $\pi(P^m)/(1-\delta)$, the sequence of lower bounds converges. Call $\overline{V}$ the value to which it converges.

Since $\mathcal{P}^t$ is nondecreasing and bounded, it converges; let $\mathcal{P}^\infty \equiv \lim_{t\to\infty}\mathcal{P}^t$. Thus, $\overline{V} = \pi(\mathcal{P}^\infty)/(1-\delta)$. An OSSPE price path is bounded from above by $\mathcal{P}^\infty$. If it does not converge to $\mathcal{P}^\infty$, then V^t is bounded below $\pi(\mathcal{P}^t)/(1-\delta)$ as $t\to\infty$, but this contradicts $\pi(\mathcal{P}^t)/(1-\delta)$ being a lower bound on the value function. Therefore, an OSSPE price path must converge to $\mathcal{P}^\infty$. For incentive compatibility to hold, it must then be true that

$$\lim_{t\to\infty}\left[\frac{\pi\left(\mathcal{P}^t\right)}{1-\delta} - \Lambda\left(\mathcal{P}^t\right)\right] \geq 0. \tag{A27}$$

By the definition of P^* being the highest constant price path that is IC in the steady state (that is, with damages equal to their steady-state value of $\gamma x(P^*)/(1-\beta)$), it follows from (A27) that $\mathcal{P}^\infty \leq P^*$. The final step is to show $\mathcal{P}^\infty = P^*$.

If $\mathcal{P}^\infty < P^*$, then

$$\lim_{t\to\infty}\left[\frac{\pi\left(\overline{P}^t\right)}{1-\delta} - \Lambda\left(\overline{P}^t\right)\right] > 0.$$

Recall that the cartel payoff is

$$\sum_{t=1}^{\infty}\delta^{t-1}\prod_{j=1}^{t-1}\left[1-\phi(P^j, P^{j-1})\right]\pi(P^t)$$
$$+\sum_{t=1}^{\infty}\delta^t\phi(P^t, P^{t-1})\prod_{j=1}^{t-1}\left[1-\phi(P^j, P^{j-1})\right]\left[\frac{\hat{\pi}}{1-\delta} - \beta^t X^0 - \sum_{j=1}^{t}\beta^{t-j}\gamma x(P^j) - F\right].$$

Taking the derivative of it with respect to $P^{t'}$ and evaluating it at $P^{t'} = \overline{P}^{t'}$, if the ICC is not binding at t', then optimality requires that

$$\pi'\left(\overline{P}^{t'}\right) + \delta\frac{\partial\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)}{\partial P^t}\left(\frac{\hat{\pi}}{1-\delta} - \beta^{t'}X^0 - \sum_{j=1}^{t'}\beta^{t'-j}\gamma x\left(\overline{P}^j\right) - F\right)$$
$$+\delta^2\left[\frac{\partial\phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)}{\partial P^{t-1}}\left(1-\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right) - \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\frac{\partial\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)}{\partial P^t}\right]$$
$$\times\left[\frac{\hat{\pi}}{1-\delta} - \beta^{t'+1}X^0 - \sum_{j=1}^{t'+1}\beta^{t'+1-j}\gamma x\left(\overline{P}^j\right) - F\right]$$
$$-\left[\frac{\partial\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)}{\partial P^t}\left(1-\phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)\right) + \left(1-\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\right)\frac{\partial\phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)}{\partial P^{t-1}}\right]$$
$$\times\sum_{t=t'+2}^{\infty}\delta^{t-t'+1}\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)\prod_{j=t'+2}^{t-1}\left[1-\phi\left(\overline{P}^j, \overline{P}^{j-1}\right)\right]\left[\frac{\hat{\pi}}{1-\delta} - \beta^t X^0 - \sum_{j=1}^{t}\beta^{t-j}\gamma x\left(\overline{P}^j\right) - F\right]$$

$$
\begin{aligned}
&- \sum_{t=t'}^{\infty} \delta^{t-t'+1} \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) \prod_{j=t'}^{t-1} \left[1 - \phi\left(\overline{P}^{j}, \overline{P}^{j-1}\right)\right] \beta^{t-t'} \gamma x' \left(\overline{P}^{t'}\right) \\
&= 0. \qquad \text{(A28)}
\end{aligned}
$$

As $t' \to \infty$, $(\overline{P}^{t'} - \overline{P}^{t'-1}) \to 0$, which implies, by Assumptions D1–D2, that

$$
\begin{aligned}
\phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right) &\to 0, \\
\phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right) &\to 0, \\
\frac{\partial \phi\left(\overline{P}^{t'}, \overline{P}^{t'-1}\right)}{\partial P^{t}} &\to 0, \\
\frac{\partial \phi\left(\overline{P}^{t'+1}, \overline{P}^{t'}\right)}{\partial P^{t-1}} &\to 0.
\end{aligned}
$$

Thus, (A28) implies $\lim_{t' \to \infty} \pi'(\overline{P}^{t'}) = 0$. However, $P^* \leq P^m$ and, by supposition, $\lim_{t \to \infty} \overline{P}^{t} < P^*$, so that $\lim_{t' \to \infty} \pi'(\overline{P}^{t'}) > 0$. This contradiction proves that our original claim that $\mathcal{P}^{\infty} < P^*$ is false. I conclude that $\lim_{t \to \infty} \overline{P}^{t} = P^*$. *Q.E.D.*